Locally Anisotropic Structures and Nonlinear Connections in Einstein and Gauge Gravity

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Abstract

We analyze local anisotropies induced by anholonomic frames and associated nonlinear connections in general relativity and extensions to affine–Poincaré and de Sitter gauge gravity and different types of Kaluza–Klein theories. We construct some new classes of cosmological solutions of gravitational field equations describing Friedmann–Robertson–Walker like universes with rotation (ellongated and flattened) ellipsoidal or torus symmetry.

1 Introduction

The search for exact solutions with generic local anisotropy in general relativity, gauge gravity and non–Riemannian extensions has its motivation from low energy limits in modern string and Kaluza–Klein theories. Such classes of solutions constructed by using moving anholonomic frame fields (tetrads, or vierbeins; we shall use the term frames for higher dimensions) reflect a new type of constrained dynamics and locally anisotropic interactions of gravitational and matter fields [31, 32].

What are the requirements of such constructions and their physical treatment? We belive that such solutuions should have the properties: (i) they satisfy the Einstein equations in general relativity and are locally anisotropic generalizations of some known solutions in isotropic limits with a well posed Cauchy problem; (ii) the corresponding geometrical and physical values are defined, as a rule, with respect to an anholonomic system of reference which reflects the imposed constraints and supposed symmetry of locally anisotropic interactions;

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the reformulation of results for a coordinate frame is also possible; (iii) by applying the method of moving frames of reference, we can generalize the solutions to some analogous in metric–affine and/or gauge gravity, in higher dimension and string theories.

Comparing with the previos results [26, 28, 29, 30, 33] on definition of self–consistent field theories incorporating various possible anisotropic, inhomogeneous and stochastic manifestations of classical and quantum interactions on locally anisotropic and higher order anisotropic spaces, we emphasize that, in this paper, we shall be interested not in some extensions of the well known gravity theories with locally isotropic spacetimes ((pseudo)Riemannian or Riemanian–Cartan–Weyl ones, in brief, RCW–spacetimes) to Finsler geometry and its generalizations, but in a proof that locally anisotropic structures (Finsler, Lagrange and higher order developments [10, 5, 25, 13, 1, 17, 15, 3, 9]) could be induced by anholonomic frames on locally isotropic spaces, even in general relativity and its metric–affine and gauge like modifications [8, 34, 23, 24, 14, 6, 33, 22, 35].

To evolve some new (frame anholonomy) features of locally isotropic gravity theories we shall apply the methods of the geometry of anholonomic frames and associated nonlinear connection (in brief, N–connection) structures elaborated in details for bundle spaces and generalized Finsler spaces in monographs [17, 15, 3] with further developments for spinor differential geometry, superspaces and stochastic calculus in [26, 28, 29, 30]. The first rigurous global definition of N–connections is due to W. Barthel [2] but the idea and some rough constructions could be found in the E. Cartan's works [5]. We note that the point of this paper is to emphasize the generic locally anisotropic geometry and physics and apply the N–connection method for non–Finslerian (pseudo) Riemannian and RCW spacetimes. Here, it should be mentioned that anholonomic frames are considered in detail, for instance, in monographs [7, 19, 21] and with respect to geometrization of gauge theories in [14, 22] but not concerning the topic on associated N–connection structures which grounds our geometric approach to anisotropies in physical theories and developing of a new method of integrating gravitational field equations.

The paper is organized as follows: Section 2 contains a brief introduction into the geometry of anholonomic frames and associated nonlinear connection structures in (pseudo) Riemannian spaces. Section 3 is devoted to the higher order anisotropic structures in Einstein gravity. In Section 4 we formulate the theory of gauge (Yang-Mills) fields on higher order anisotropic spaces; the variational proof of gauge field equations is considered in connection with a "pure" geometrical method of definition of field equations. In Section 5 the higher order anisotropic gravity is reformulated as a gauge theory for nonsemisimple groups. A model of nonlinear de Sitter gauge gravity with higher order anisotropy is formulated in Section 6. An ansatz for generation of four dimenisonal solutions with generic anisotropy of the Einstein equations is analyzed in Section 7. Some classes of solutions, with generic anisotropy, of Einstein equations describing Friedmann–Robertson–Walker like universes with rotation (ellongated and flattened) ellipsoid and torus symmetry are constructed in Section 8. Concluding remarks are given in Section 9.

2 Anholonomic Frames on (Pseudo) Riemannian Spaces

For definiteness, we consider a $(n+m)$–dimensional (pseudo) Riemannian spacetime $V^{(n+m)}$, being a paracompact and connected Hausdorff C^∞–manifold, enabled with a nonsigular metric

$$ds^2 = \widetilde{g}_{\alpha\beta}\ du^\alpha \otimes du^\beta$$

with the coefficients

$$\widetilde{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{array} \right] \quad (1)$$

parametrized with respect to a local coordinate basis $du^\alpha = \left(dx^i, dy^a\right)$, having its dual $\partial/u^\alpha = \left(\partial/x^i, \partial/y^a\right)$, where the indices of geometrical objects and local coordinate $u^\alpha = \left(x^k, y^a\right)$ run correspondingly the values: (for Greek indices)$\alpha, \beta, \ldots = n+m$; for (Latin indices) $i, j, k, ... = 1, 2, ..., n$ and $a, b, c, ... = 1, 2, ..., m$. We shall use 'tilds' if would be necessary to emphasize that a value is defined with respect to a coordinate basis.

The metric (1) can be rewritten in a block $(n \times n) + (m \times m)$ form

$$g_{\alpha\beta} = \left(\begin{array}{cc} g_{ij}(x^k, y^a) & 0 \\ 0 & h_{ab}(x^k, y^a) \end{array} \right) \quad (2)$$

with respect to a subclass of $n+m$ anholonomic frame basis (for four dimensions one used terms tetrads, or vierbiends) defined

$$\delta_\alpha = (\delta_i, \partial_a) = \frac{\delta}{\partial u^\alpha} = \left(\delta_i = \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^b\left(x^j, y^c\right) \frac{\partial}{\partial y^b}, \partial_a = \frac{\partial}{\partial y^a} \right) \quad (3)$$

and

$$\delta^\beta = \left(d^i, \delta^a\right) = \delta u^\beta = \left(d^i = dx^i, \delta^a = \delta y^a = dy^a + N_k^a\left(x^j, y^b\right) dx^k \right), \quad (4)$$

called the locally anisotropic bases (in brief, la–bases) adapted to the coefficients N_j^a. The $n \times n$ matrice g_{ij} defines the so–called horizontal metric (in brief, h–metric) and the $m \times m$ matrice h_{ab} defines the vertical (v–metric) with respect to the associated nonlinear connection (N–connection) structure given by its coefficients $N_j^a\left(u^\alpha\right)$ from (3) and (4). The geometry of N–connections is studied in detail in [2, 17]; here we shall consider its applications with respect to anholonomic frames in general relativity and its locally isotropic generalizations.

A frame structure δ_α (3) on $V^{(n+m)}$ is characterized by its anholonomy relations

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w^\gamma{}_{\alpha\beta} \delta_\gamma. \quad (5)$$

with anholonomy coefficients $w^\alpha{}_{\beta\gamma}$.The elongation of partial derivatives (by N–coefficients) in the locally adapted partial derivatives (3) reflects the fact that on the (pseudo) Riemannian spacetime $V^{(n+m)}$ it is modelled a generic local

anisotropy characterized by the anholonomy relations (5) when the anholonomy coefficients are computed as follows

$$\begin{aligned} w^k{}_{ij} &= 0, w^k{}_{aj} = 0, w^k{}_{ia} = 0, w^k{}_{ab} = 0, w^c{}_{ab} = 0, \\ w^a{}_{ij} &= -\Omega^a_{ij}, w^b{}_{aj} = -\partial_a N^b_i, w^b{}_{ia} = \partial_a N^b_i, \end{aligned}$$

where

$$\Omega^a_{ij} = \partial_i N^a_j - \partial_j N^a_i + N^b_i \partial_b N^a_j - N^b_j \partial_b N^a_i$$

defines the coefficients of the N–connection curvature, in brief, N–curvature. On (pseudo) Riemannian spacetimes this is a characteristic of a chosen anholonomic system of reference.

A N–connection N defines a global decomposition,

$$N : V^{(n+m)} = H^{(n)} \oplus V^{(m)},$$

of spacetime $V^{(n+m)}$ into a n–dimensional horizontal subspace $H^{(n)}$ (with holonomic x–coordinates) and into a m–dimensional vertical subspace $V^{(m)}$ (with anisotropic, anholonomic, y–coordinates). This form of parametrizations of sets of mixt holonomic–anholonomic frames is very useful for investigation, for instance, of kinetic and thermodynamic systems in general relativity, spinor and gauge field interactions in curved spacetimes and for definition of non–trivial reductions from higher dimension to lower dimension ones in Kaluza–Klein theories. In the last case the N–connection could be treated as a 'splitting' field into base's and extra dimensions with the anholonomic (equivalently, anisotropic) structure defined from some prescribed types of symmetries and constraints (imposed on a physical system) or, for a different class of theories, with some dynamical field equations following in the low energy limit of string theories [27, 28] or from Einstein equations on a higher dimension space.

The locally anisotropic spacetimes, la–spacetimes, to be investigated in this section are considered to be some (pseudo) Riemannian manifolds $V^{(n+m)}$ enabled with a frame, in general, anholonomic structures of basis vector fields, $\delta^\alpha = (\delta^i, \delta^a)$ and theirs duals $\delta_\alpha = (\delta_i, \delta_a)$ (equivalently to an associated N–connection structure), adapted to a symmetric metric field $g_{\alpha\beta}$ (2) of necessary signature and to a linear, in general nonsymmetric, connection $\Gamma^\alpha{}_{\beta\gamma}$ defining the covariant derivation D_α satisfying the metricity conditions $D_\alpha g_{\beta\gamma} = 0$. The term la– points to a prescribed type of anholonomy structure. As a matter of principle, on a (pseudo) Riemannian spacetime, we can always, at least locally, remove our considerations with respect to a coordinate basis. In this case the geometric anisotopy is modelled by metrics of type (1). Such ansatz for metrics are largely applied in modern Kaluza–Klein theory [20] where the N–conection structures have been not pointed out because in the simplest approximation on topological compactification of extra dimensions the N–connection geometry is trivial. A rigorous anlysis of systems with mixed holonomic–anholonomic variables was not yet provided for general relativity, extra dimension and gauge like gravity theories..

A $n + m$ anholonomic structure distinguishes (d) the geometrical objects into h– and v–components. Such objects are briefly called d–tensors, d–metrics and/or d–connections. Their components are defined with respect to a la–basis of

type (3), its dual (4), or their tensor products (d–linear or d–affine transforms of such frames could also be considered). For instance, a covariant and contravariant d–tensor Z, is expressed

$$Z = Z^{\alpha}_{\ \beta}\delta_\alpha \otimes \delta^\beta = Z^{i}_{\ j}\delta_i \otimes d^j + Z^{i}_{\ a}\delta_i \otimes \delta^a + Z^{b}_{\ j}\partial_b \otimes d^j + Z^{b}_{\ a}\partial_b \otimes \delta^a. \qquad (6)$$

A linear d–connection D on la–space $V^{(n+m)}$,

$$D_{\delta_\gamma}\delta_\beta = \Gamma^{\alpha}_{\ \beta\gamma}\left(x,y\right)\delta_\alpha,$$

is parametrized by non–trivial h–v–components,

$$\Gamma^{\alpha}_{\ \beta\gamma} = \left(L^{i}_{\ jk}, L^{a}_{\ bk}, C^{i}_{\ jc}, C^{a}_{\ bc}\right). \qquad (7)$$

A metric on $V^{(n+m)}$ with $(m\times m)+(n\times n)$ block coefficients (2) is written in distinguished form, as a metric d–tensor (in brief, d–metric), with respect to a la–base (4)

$$\delta s^2 = g_{\alpha\beta}\left(u\right)\delta^\alpha \otimes \delta^\beta = g_{ij}(x,y)dx^i dx^j + h_{ab}(x,y)\delta y^a \delta y^b. \qquad (8)$$

Some d–connection and d–metric structures are compatible if there are satisfied the conditions

$$D_\alpha g_{\beta\gamma} = 0.$$

For instance, a canonical compatible d–connection

$$^{c}\Gamma^{\alpha}_{\ \beta\gamma} = \left(^{c}L^{i}_{\ jk}, ^{c}L^{a}_{\ bk}, ^{c}C^{i}_{\ jc}, ^{c}C^{a}_{\ bc}\right)$$

is defined by the coefficients of d–metric (8), $g_{ij}\left(x,y\right)$ and $h_{ab}\left(x,y\right)$, and by the N–coefficients,

$$\begin{aligned}
^{c}L^{i}_{\ jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \qquad (9)\\
^{c}L^{a}_{\ bk} &= \partial_b N^a_k + \frac{1}{2}h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N^d_i - h_{db}\partial_c N^d_i\right),\\
^{c}C^{i}_{\ jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk},\\
^{c}C^{a}_{\ bc} &= \frac{1}{2}h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right)
\end{aligned}$$

The coefficients of the canonical d–connection generalize for la–spacetimes the well known Cristoffel symbols; on a (pseudo) Riemannian spacetime with a fixed anholonomic frame the d–connection coefficients transform exactly into the metric connection coefficients.

For a d–connection (7) the components of torsion,

$$\begin{aligned}
T\left(\delta_\gamma, \delta_\beta\right) &= T^{\alpha}_{\beta\gamma}\delta_\alpha,\\
T^{\alpha}_{\beta\gamma} &= \Gamma^{\alpha}_{\ \beta\gamma} - \Gamma^{\alpha}_{\ \gamma\beta} + w^{\alpha}_{\ \beta\gamma}
\end{aligned}$$

are expressed via d–torsions

$$\begin{aligned}
T^{i}_{.jk} &= -T^{i}_{.kj} = L^{i}_{jk} - L^{i}_{kj}, \quad T^{i}_{ja} = C^{i}_{.ja}, T^{i}_{aj} = -C^{i}_{ja},\\
T^{i}_{.ab} &= 0, \quad T^{a}_{.bc} = S^{a}_{.bc} = C^{a}_{bc} - C^{a}_{cb}, \qquad (10)\\
T^{a}_{.ij} &= -\Omega^{a}_{ij}, \quad T^{a}_{.bi} = \partial_b N^a_i - L^{a}_{.bj}, \quad T^{a}_{.ib} = -T^{a}_{.bi}.
\end{aligned}$$

We note that for symmetric linear connections the d–torsions are induced as a pure anholonomic effect. They vanish with respect to a coordinate frame of reference.

In a similar manner, putting non–vanishing coefficients (7) into the formula for curvature,

$$\begin{aligned} R\left(\delta_\tau,\delta_\gamma\right)\delta_\beta &= R_{\beta\ \gamma\tau}^{\ \alpha}\delta_\alpha, \\ R_{\beta\ \gamma\tau}^{\ \alpha} &= \delta_\tau\Gamma^\alpha_{\ \beta\gamma}-\delta_\gamma\Gamma^\alpha_{\ \beta\delta}+\Gamma^\varphi_{\ \beta\gamma}\Gamma^\alpha_{\ \varphi\tau}-\Gamma^\varphi_{\ \beta\tau}\Gamma^\alpha_{\ \varphi\gamma}+\Gamma^\alpha_{\ \beta\varphi}w^\varphi_{\ \gamma\tau}, \end{aligned}$$

we can compute the components of d–curvatures

$$\begin{aligned} R_{h.jk}^{.i} &= \delta_k L^i_{.hj}-\delta_j L^i_{.hk}+L^m_{.hj}L^i_{mk}-L^m_{.hk}L^i_{mj}-C^i_{.ha}\Omega^a_{.jk}, \\ R_{b.jk}^{.a} &= \delta_k L^a_{.bj}-\delta_j L^a_{.bk}+L^c_{.bj}L^a_{.ck}-L^c_{.bk}L^a_{.cj}-C^a_{.bc}\Omega^c_{.jk}, \\ P_{j.ka}^{.i} &= \partial_k L^i_{.jk}+C^i_{.jb}T^b_{.ka}-(\partial_k C^i_{.ja}+L^i_{.lk}C^l_{.ja}-L^l_{.jk}C^i_{.la}-L^c_{.ak}C^i_{.jc}), \\ P_{b.ka}^{.c} &= \partial_a L^c_{.bk}+C^c_{.bd}T^d_{.ka}-(\partial_k C^c_{.ba}+L^c_{.dk}C^d_{.ba}-L^d_{.bk}C^c_{.da}-L^d_{.ak}C^c_{.bd}), \\ S_{j.bc}^{.i} &= \partial_c C^i_{.jb}-\partial_b C^i_{.jc}+C^h_{.jb}C^i_{.hc}-C^h_{.jc}C^i_{hb}, \\ S_{b.cd}^{.a} &= \partial_d C^a_{.bc}-\partial_c C^a_{.bd}+C^e_{.bc}C^a_{.ed}-C^e_{.bd}C^a_{.ec}. \end{aligned}$$

The Ricci tensor

$$R_{\beta\gamma}=R_{\beta\ \gamma\alpha}^{\ \alpha}$$

has the d–components

$$\begin{aligned} R_{ij} &= R_{i.jk}^{.k}, \quad R_{ia}=-{}^2P_{ia}=-P_{i.ka}^{.k}, \\ R_{ai} &= {}^1P_{ai}=P_{a.ib}^{.b}, \quad R_{ab}=S_{a.bc}^{.c}. \end{aligned} \tag{11}$$

We point out that because, in general, ${}^1P_{ai}\neq\ {}^2P_{ia}$, the Ricci d-tensor is non symmetric.

Having defined a d-metric of type (8) in $V^{(n+m)}$ we can compute the scalar curvature

$$\overleftarrow{R}=g^{\beta\gamma}R_{\beta\gamma}$$

of a d-connection D,

$$\overleftarrow{R}=\widehat{R}+S, \tag{12}$$

where $\widehat{R}=g^{ij}R_{ij}$ and $S=h^{ab}S_{ab}$.

Now, by introducing the values (11) and (12) into the Einstein's equations

$$R_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}\overleftarrow{R}=k\Upsilon_{\beta\gamma},$$

we can write down the system of field equations for la–gravity with anholonomic (N–connection) structure:

$$\begin{aligned} R_{ij}-\frac{1}{2}\left(\widehat{R}+S\right)g_{ij} &= k\Upsilon_{ij}, \\ S_{ab}-\frac{1}{2}\left(\widehat{R}+S\right)h_{ab} &= k\Upsilon_{ab}, \\ {}^1P_{ai} &= k\Upsilon_{ai}, \\ {}^2P_{ia} &= -k\Upsilon_{ia}, \end{aligned} \tag{13}$$

where $\Upsilon_{ij}, \Upsilon_{ab}, \Upsilon_{ai}$ and Υ_{ia} are the components of the energy–momentum d–tensor field $\Upsilon_{\beta\gamma}$ (which includes possible cosmological constants, contributions of anholonomy d–torsions (10) and matter) and k is the coupling constant.

The h- v- decomposition of gravitational field equations (13) was introduced by Miron and Anastasiei [17] in their N–connection approach to generalized Finsler and Lagrange spaces. It holds true as well on (pseudo) Riemannian spaces, in general gravity; in this case we obtain the usual form of Einstein equations if we transfer considerations with respect to coordinate frames. If the N–coefficients are prescribed by fixing the anholonomic frame of reference, different classes of solutions are to be constructed by finding the h– and v–components, g_{ij} and h_{ab}, of metric (1), or its equivalent (2). A more general approach is to consider the N–connection as 'free' but subjected to the condition that its coefficients along with the d–metric components are chosen to solve the Einsten equations in the form (13) for some suggested symmetries, configurations of horizons and type of singularities and well defined Cauchy problem. This way one can construct new classes of metrics with generic local anisotropy (see [31] and [32] and Sections 7 in this paper).

3 Higher Order Anisotropic Structures

Miron and Atanasiu [18, 15, 16] developed the higher order Lagrange and Finsler geometry with applications in mechanics in order to geometrize the concepts of classical mechanics on higher order tangent bundles. The work [28] was a proof that higher order anisotropies (in brief, one writes abbreviations like ha–, ha–superspace, ha–spacetime, ha–geometry and so on) can be induced alternatively in low energy limits of (super) string theories and a higher order superbundle N–connection formalism was proposed. There were developed the theory of spinors [29], proposed models of ha–(super)gravity and matter interactions on ha–spaces and defined the supersymmetric stochastic calculus in ha–superspaces which were summarized in the monograph [30] containing a local (super) geometric approach to so called ha–superstring and generalized Finsler–Kaluza–Klein (super) gravities.

The aim of this section is to proof that higher order anisotropic (ha–structures) are induced by respective anholonomic frames in higher dimension Einstein gravity, to present the basic geometric background for a such moving frame formalism and associated N–connections and to deduce the system of gravitational field equations with respect to ha–frames.

3.1 Ha–frames and corresponding N–connections

Let us consider a (pseudo) Riemannian spacetime $V^{(\overline{n})} = V^{(n+\overline{m})}$ where the anisotropic dimension $\overline{m}$ is split into z sub–dimensions m_p, $(p = 1, 2, ..., z)$, i. e. $\overline{m} = m_1 + m_2 + ... + m_z$. The local coordinates on a such higher dimension curved spacetime will be denoted as to take into account the m–decomposition,

$$\begin{aligned} u &= \{u^{\overline{\alpha}} \equiv u^{\alpha_z} = (x^i, y^{a_1}, y^{a_2}, \ldots, y^{a_p}, \ldots y^{a_z})\}, \\ u^{\alpha_p} &= (x^i, y^{a_1}, y^{a_2}, \ldots, y^{a_p}) = (u^{\alpha_{p-1}}, y^{a_p}) \,. \end{aligned}$$

The la–constructions from the previous Section are considered to describe anholonomic structures of first order; for $z=1$ we put $u^{\alpha_1}=(x^i,y^{a_1})=u^{\alpha}=(x^i,y^{a_1})$. The higher order anisotropies are defined inductively, 'shell by shell', starting from the first order to the higher order, z–anisotropy. In order to distinguish the components of geometrical objects with respect to a p–shell we provide both Greek and Latin indices with a corresponding subindex like $\alpha_p=(\alpha_{p-1},a_p)$, and $a_p=(1,2,...,m_p)$, i. e. one holds a shell parametrization for coordinates,

$$y^{a_p}=(y^1_{(p)}=y^1,y^2_{(p)}=y^2,...,y^{m_p}_{(p)}=y^{m_p}).$$

We shall overline some indices, for instance, $\overline{\alpha}$ and $\overline{a}$, if would be necessary to point that it could be split into shell components and omit the p–shell mark (p) if this does not lead to misunderstanding. Such decompositions of indices and geometrical and physical values are introduced with the aim for a further modelling of (in general, dynamical) spllittings of higher dimension spacetimes, step by step, with 'interior' subspaces being of different dimension, to lower dimensions, with nontrival topology and anholonomic (anisotropy) structures in generalized Kaluza–Klein theories.

The coordinate frames are denoted

$$\partial_{\overline{\alpha}}=\partial/u^{\overline{\alpha}}=\left(\partial/x^i,\partial/y^{a_1},...,\partial/y^{a_z}\right)$$

with the dual ones

$$d^{\overline{\alpha}}=du^{\overline{\alpha}}=\left(dx^i,dy^{a_1},...,dy^{a_z}\right),$$

when

$$\partial_{\alpha_p}=\partial/u^{\alpha_p}=\left(\partial/x^i,\partial/y^{a_1},...,\partial/y^{a_p}\right)$$

and

$$d^{\alpha_p}=du^{\alpha_p}=\left(dx^i,dy^{a_1},...,dy^{a_p}\right)$$

if considerations are limited to the p-th shell.

With respect to a coordinate frame a nonsigular metric

$$ds^2=\widetilde{g}_{\overline{\alpha}\overline{\beta}}\,du^{\overline{\alpha}}\otimes du^{\overline{\beta}}$$

with coefficients $\widetilde{g}_{\overline{\alpha}\overline{\beta}}$ defined on induction,

$$\begin{aligned}
\widetilde{g}_{\alpha_1\beta_1} &= \left[\begin{array}{cc} g_{ij}+M_i^{a_1}M_j^{b_1}h_{a_1b_1} & M_j^{e_1}h_{a_1e_1} \\ M_i^{e_1}h_{b_1e_1} & h_{a_1b_1}\end{array}\right], \qquad (14)\\
&\vdots \\
\widetilde{g}_{\alpha_p\beta_p} &= \left[\begin{array}{cc} g_{\alpha_{p-1}\beta_{p-1}}+M_{\alpha_{p-1}}^{a_p}M_{\beta_{p-1}}^{b_p}h_{a_pb_p} & M_{\beta_{p-1}}^{e_p}h_{a_pe_p} \\ M_{\alpha_{p-1}}^{e_p}h_{b_pe_p} & h_{a_pb_p}\end{array}\right], \\
&\vdots \\
\widetilde{g}_{\overline{\alpha}\overline{\beta}}=\widetilde{g}_{\alpha_z\beta_z} &= \left[\begin{array}{cc} g_{\alpha_{z-1}\beta_{z-1}}+M_{\alpha_{z-1}}^{a_z}M_{\beta_{z-1}}^{b_z}h_{a_zb_z} & M_{\beta_{z-1}}^{e_z}h_{a_ze_z} \\ M_{\alpha_{z-1}}^{e_z}h_{b_ze_z} & h_{a_zb_z}\end{array}\right],
\end{aligned}$$

where indices are split as $\alpha_1=(i_1,a_1)$, $\alpha_2=(\alpha_1,a_2)$, $\alpha_p=(\alpha_{p-1},a_p)$; $p=1,2,...z$.

The metric (14) on $V^{(\overline{n})}$ splits into symmetric blocks of matrices of dimensions

$$(n\times n)\oplus(m_1\times m_1)\oplus...\oplus(m_z\times m_z)\,,$$

$n+m$ form

$$g_{\alpha\beta}=\begin{pmatrix} g_{ij}(u) & 0 & \dots & 0\\ 0 & h_{a_1b_1} & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & h_{a_zb_z}\end{pmatrix} \tag{15}$$

with respect to an anholonomic frame basis defined on induction

$$\begin{aligned} \delta_{\alpha_p} &= (\delta_{\alpha_{p-1}},\partial_{a_p})=(\delta_i,\delta_{a_1},...,\delta_{a_{p-1}},\partial_{a_p}) \\ &= \frac{\delta}{\partial u^{\alpha_p}}=\left(\frac{\delta}{\partial u^{\alpha_{p-1}}}=\frac{\partial}{\partial u^{\alpha_{p-1}}}-N_{\alpha_{p-1}}^{b_p}(u)\frac{\partial}{\partial y^{b_p}},\frac{\partial}{\partial y^{a_p}}\right), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \delta^{\beta_p} &= \left(d^i,\delta^{\overline{a}_p}\right)=\left(d^i,\delta^{a_1},...,\delta^{a_{p-1}},\delta^{a_p}\right) \\ &= \delta u^{\beta_p}=\left(d^i=dx^i,\delta^{\overline{a}_p}=\delta y^{\overline{a}_p}=dy^{\overline{a}_p}+M_{\alpha_{p-1}}^{\overline{a}_p}(u)\,du^{\alpha_{p-1}}\right), \end{aligned} \tag{17}$$

where $\overline{a}_p=(a_1,a_2,...,a_p)$, are called the locally anisotropic bases (in brief la–bases) adapted respectively to the N–coefficients

$$N_{\alpha_{p-1}}^{a_p}=\left\{N_i^{a_p},N_{a_1}^{a_p},...,N_{a_{p-2}}^{a_p},N_{a_{p-1}}^{a_p}\right\}$$

and M–coefficients

$$M_{\alpha_{p-1}}^{a_p}=\left\{M_i^{a_p},M_{a_1}^{a_p},...,M_{a_{p-2}}^{a_p},M_{a_{p-1}}^{a_p}\right\};$$

the coefficients $M_{\alpha_{p-1}}^{a_p}$ are related via some algebraic relations with $N_{\alpha_{p-1}}^{a_p}$ in order to be satisfied the la–basis duality conditions

$$\delta_{\alpha_p}\otimes\delta^{\beta_p}=\delta_{\alpha_p}^{\beta_p},$$

where $\delta_{\alpha_p}^{\beta_p}$ is the Kronecker symbol, for every shell.

The geometric structure of N– and M–coefficients of a higher order nonlinear connection becames more explicit if we write the relations (16) and (17) in matrix form, respectively,

$$\delta_\bullet=\widehat{N}(u)\times\partial_\bullet$$

and

$$\delta^\bullet=d^\bullet\times M(u)\,,$$

where

$$\delta_\bullet=\delta_{\overline{\alpha}}=\begin{pmatrix}\delta_i\\ \delta_{a_1}\\ \delta_{a_2}\\ \dots\\ \delta_{a_z}\end{pmatrix}=\begin{pmatrix}\delta/\partial x^i\\ \delta/\partial y^{a_1}\\ \delta/\partial y^{a_2}\\ \dots\\ \delta/\partial y^{a_z}\end{pmatrix},\partial_\bullet=\partial_{\overline{\alpha}}=\begin{pmatrix}\partial_i\\ \partial_{a_1}\\ \partial_{a_2}\\ \dots\\ \partial_{a_z}\end{pmatrix}=\begin{pmatrix}\partial/\partial x^i\\ \partial/\partial y^{a_1}\\ \partial/\partial y^{a_2}\\ \dots\\ \partial/\partial y^{a_z}\end{pmatrix},$$

$$\delta^{\bullet} = \begin{pmatrix} dx^i & \delta y^{a_1} & \delta y^{a_2} & \ldots & \delta y^{a_z} \end{pmatrix}, \; d^{\bullet} = \begin{pmatrix} dx^i & dy^{a_1} & dy^{a_2} & \ldots & dy^{a_z} \end{pmatrix},$$

and

$$\widehat{N} = \begin{pmatrix} 1 & -N_i^{a_1} & -N_i^{a_2} & \ldots & -N_i^{a_z} \\ 0 & 1 & -N_{a_1}^{a_2} & \ldots & -N_{a_1}^{a_z} \\ 0 & 0 & 1 & \ldots & -N_{a_2}^{a_z} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & M_i^{a_1} & M_i^{a_2} & \ldots & M_i^{a_z} \\ 0 & 1 & M_{a_1}^{a_2} & \ldots & M_{a_1}^{a_z} \\ 0 & 0 & 1 & \ldots & M_{a_2}^{a_z} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

The $n \times n$ matrice g_{ij} defines the horizontal metric (in brief, h–metric) and the $m_p \times m_p$ matrices $h_{a_p b_p}$ defines the vertical, v_p–metrics with respect to the associated nonlinear connection (N–connection) structure given by its coefficients $N^{a_p}_{\alpha_{p-1}}$ from (16). The geometry of N–connections on higher order tangent bundles is studied in detail in [18, 15, 16], for vector (super)bundles there it was proposed the approach from [28, 30]; the approach and denotations elaborated in this work is adapted to further applications in higher dimension Einstein gravity and its non–Riemannian locally anisotropic extensions.

A ha–basis $\delta_{\overline{\alpha}}$ (4) on $V^{(\overline{n})}$ is characterized by its anholonomy relations

$$\delta_{\overline{\alpha}}\delta_{\overline{\beta}} - \delta_{\overline{\beta}}\delta_{\overline{\alpha}} = w^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}\delta_{\overline{\gamma}}. \tag{18}$$

with anholonomy coefficients $w^{\overline{\gamma}}{}_{\overline{\alpha}\overline{\beta}}$. The anholonomy coefficients are computed

$$\begin{aligned} w^k{}_{ij} &= 0; w^k{}_{a_p j} = 0; w^k{}_{i a_p} = 0; w^k{}_{a_p b_p} = 0; w^{c_p}{}_{a_p b_p} = 0; \\ w^{a_p}{}_{ij} &= -\Omega^{a_p}_{ij}; w^{b_p}{}_{a_p j} = -\delta_{a_p} N_i^{b_p}; w^{b_p}{}_{i a_p} = \delta_{a_p} N_i^{b_p}; \\ w^{k_p}{}_{a_p b_p} &= 0; w^{c_f}{}_{a_p b_f} = 0, f < p; w^{c_f}{}_{b_f a_p} = 0, f < p; w^{c_f}{}_{a_p b_p} = 0, f < p; \\ w^{a_p}{}_{c_f d_s} &= -\Omega^{a_p}_{c_f d_s}, (f, s < p); w^{b_p}{}_{a_p c_f} = -\delta_{a_p} N^{b_p}_{c_f}, f < p; w^{b_p}{}_{c_f a_p} = \delta_{a_p} N^{b_p}_{c_f}, f < p; \end{aligned}$$

where

$$\begin{aligned} \Omega^{a_p}_{ij} &= \partial_i N_j^{a_p} - \partial_j N_i^{a_p} + N_i^{b_p}\delta_{b_p} N_j^{a_p} - N_j^{b_p}\delta_{b_p} N_i^{a_p}, \\ \Omega^{a_p}_{\alpha_f \beta_s} &= \partial_{\alpha_f} N^{a_p}_{\beta_s} - \partial_{\beta_s} N^{a_p}_{\alpha_f} + N^{b_p}_{\alpha_f}\delta_{b_p} N^{a_p}_{\beta_s} - N^{b_p}_{\beta_s}\delta_{b_p} N^{a_p}_{\alpha_f}, \end{aligned} \tag{19}$$

for $1 \leq s, f < p$, are the coefficients of higher order N–connection curvature (N–curvature).

A higher order N–connection N defines a global decomposition

$$N: \; V^{(\overline{n})} = H^{(n)} \oplus V^{(m_1)} \oplus V^{(m_2)} \oplus ... \oplus V^{(m_z)},$$

of spacetime $V^{(\overline{n})}$ into a n–dimensional horizontal subspace $H^{(n)}$ (with holonomic x–components) and into m_p–dimensional vertical subspaces $V^{(m_p)}$ (with anisotropic, anholonomic, $y_{(p)}$–components).

3.2 Distinguished linear connections

In this section we consider fibered (pseudo) Riemannian manifolds $V^{(\overline{n})}$ enabled with anholonomic frame structures of basis vector fields, $\delta^{\overline{\alpha}} = (\delta^i, \delta^{\overline{a}})$ and theirs duals $\delta_{\overline{\alpha}} = (\delta_i, \delta_{\overline{a}})$ with associated N–connection structure, adapted to a symmetric metric field $g_{\overline{\alpha}\overline{\beta}}$ (15) and to a linear, in general nonsymmetric, connection $\Gamma^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}$ defining the covariant derivation $D_{\overline{\alpha}}$ satisfying the metricity conditions $D_{\overline{\alpha}} g_{\overline{\beta}\overline{\gamma}} = 0$. Such spacetimes are provided with anholonomic higher order anisotropic structures and, in brief, are called ha–spacetimes.

A higher order N–connection distinguishes (d) the geometrical objects into h– and v_p–components (d–tensors, d–metrics and/or d–connections). For instance, a d-tensor field of type $\begin{pmatrix} p & r_1 & ... & r_p & ... & r_z \\ q & s_1 & ... & s_p & ... & s_z \end{pmatrix}$ is written in local form as

$$\begin{aligned}
\mathbf{t} &= t^{i_1 ... i_p a_1^{(1)} ... a_{r_1}^{(1)} ... a_1^{(p)} ... a_{r_p}^{(p)} ... a_1^{(z)} ... a_{r_z}^{(z)}}_{j_1 ... j_q b_1^{(1)} ... b_{r_1}^{(1)} ... b_1^{(p)} ... b_{r_p}^{(p)} ... b_1^{(z)} ... b_{r_z}^{(z)}} (u)\, \delta_{i_1} \otimes ... \otimes \delta_{i_p} \otimes d^{j_1} \otimes ... \otimes d^{j_q} \otimes \\
& \delta_{a_1^{(1)}} \otimes ... \otimes \delta_{a_{r_1}^{(1)}} \otimes \delta^{b_1^{(1)}} ... \otimes \delta^{b_{s_1}^{(1)}} \otimes ... \otimes \delta_{a_1^{(p)}} \otimes ... \otimes \delta_{a_{r_p}^{(p)}} \otimes ... \otimes \\
& \delta^{b_1^{(p)}} ... \otimes \delta^{b_{s_p}^{(p)}} \otimes \delta_{a_1^{(z)}} \otimes ... \otimes \delta_{a_{r_z}^{(z)}} \otimes \delta^{b_1^{(z)}} ... \otimes \delta^{b_{s_z}^{(z)}}.
\end{aligned}$$

A linear d–connection D on ha–spacetime $V^{(\overline{n})}$,

$$D_{\delta_{\overline{\gamma}}} \delta_{\overline{\beta}} = \Gamma^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}(u)\, \delta_{\overline{\alpha}},$$

is defined by its non–trivial h–v–components,

$$\Gamma^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}} = \left(L^i_{\ jk}, L^{\overline{a}}_{\ \overline{b}k}, C^i_{\ j\overline{c}}, C^{\overline{a}}_{\ \overline{b}\overline{c}}, K^{a_p}_{\ b_p c_p}, K^{a_p}_{\ b_s c_f}, Q^{a_f}_{\ b_f c_p} \right), \tag{20}$$

for $f < p, s$.

A metric with block coefficients (15) is written as a d–metric, with respect to a la–base (17)

$$\delta s^2 = g_{\overline{\alpha}\overline{\beta}}(u)\, \delta^{\overline{\alpha}} \otimes \delta^{\overline{\beta}} = g_{ij}(u) dx^i dx^j + h_{a_p b_p}(u) \delta y^{a_p} \delta y^{b_p}, \tag{21}$$

where $p = 1, 2, ..., z$.

A d–connection and a d–metric structure are compatible if there are satisfied the conditions

$$D_{\overline{\alpha}} g_{\overline{\beta}\overline{\gamma}} = 0.$$

The canonical d–connection ${}^c\Gamma^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}$ is defined by the coefficients of d–metric (21), and by the higher order N–coefficients,

$$\begin{aligned}
{}^cL^i_{\ jk} &= \frac{1}{2} g^{in} \left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk} \right), \qquad (22) \\
{}^cL^{\overline{a}}_{\ \overline{b}k} &= \delta_{\overline{b}} N^{\overline{a}}_k + \frac{1}{2} h^{\overline{a}\overline{c}} \left(\partial_k h_{\overline{b}\overline{c}} - h_{\overline{d}\overline{c}} \delta_{\overline{b}} N^{\overline{d}}_k - h_{\overline{d}\overline{b}} \delta_{\overline{c}} N^{\overline{d}}_k \right), \\
{}^cC^i_{\ j\overline{c}} &= \frac{1}{2} g^{ik} \delta_{\overline{c}} g_{jk}, \\
{}^cC^{\overline{a}}_{\ \overline{b}\overline{c}} &= \frac{1}{2} h^{\overline{a}\overline{d}} \left(\delta_{\overline{c}} h_{\overline{d}\overline{b}} + \delta_{\overline{b}} h_{\overline{d}\overline{c}} - \delta_{\overline{d}} h_{\overline{b}\overline{c}} \right),
\end{aligned}$$

$$
\begin{aligned}
{}^{c}K^{a_p}_{\ b_pc_p} &= \frac{1}{2}g^{a_pe_p}\left(\delta_{c_p}g_{e_pb_p}+\delta_{b_p}g_{e_pc_p}-\delta_{e_p}g_{b_pc_p}\right),\\
{}^{c}K^{a_p}_{\ b_se_f} &= \delta_{b_s}N^{a_p}_{e_f}+\frac{1}{2}h^{a_pc_p}\left(\partial_{e_f}h_{b_sc_p}-h_{d_pc_p}\delta_{b_s}N^{d_p}_{e_f}-h_{d_sb_s}\delta_{c_p}N^{d_s}_{e_f}\right),\\
{}^{c}Q^{a_f}_{\ b_fc_p} &= \frac{1}{2}h^{a_fe_f}\delta_{c_p}h_{b_fe_f},
\end{aligned}
$$

where $f<p,s$. They transform into usual Christoffel symbols with respect to a coordinate base.

3.3 Ha–torsions and ha–curvatures

For a higher order anisotropic d–connection (20) the components of torsion,

$$
\begin{aligned}
T\left(\delta_{\overline{\gamma}},\delta_{\overline{\beta}}\right) &= T^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}\delta_{\overline{\alpha}},\\
T^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}} &= \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}-\Gamma^{\overline{\alpha}}_{\overline{\gamma}\overline{\beta}}+w^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}
\end{aligned}
$$

are expressed via d–torsions

$$
\begin{aligned}
T^{i}_{.jk} &= -T^{i}_{.kj}=L^{i}_{jk}-L^{i}_{kj},\quad T^{i}_{j\overline{a}}=-T^{i}_{\overline{a}j}=C^{i}_{.j\overline{a}},\\
T^{i}_{.\overline{a}\overline{b}} &= 0,\quad T^{\overline{a}}_{.\overline{b}\overline{c}}=S^{\overline{a}}_{.\overline{b}\overline{c}}=C^{\overline{a}}_{\overline{b}\overline{c}}-C^{\overline{a}}_{\overline{c}\overline{b}},\\
T^{\overline{a}}_{.ij} &= -\Omega^{\overline{a}}_{ij},\quad T^{\overline{a}}_{.\overline{b}i}=-T^{\overline{a}}_{.\overline{b}i}=\delta_{\overline{b}}N^{\overline{a}}_{i}-L^{\overline{a}}_{.\overline{b}j},\\
T^{a_f}_{.b_fc_f} &= -T^{a_f}_{.c_fb_f}=K^{a_f}_{.b_fc_f}-K^{a_f}_{.c_fb_f},\\
T^{a_f}_{.a_pb_s} &= 0,\quad T^{a_f}_{.b_fa_p}=-T^{a_f}_{.a_pb_f}=Q^{a_f}_{.b_fa_p},\\
T^{a_p}_{.a_fb_f} &= -\Omega^{a_p}_{.a_fb_f},\quad T^{a_p}_{.b_sa_f}=-T^{a_p}_{.a_fb_s}=\delta_{b_s}N^{a_p}_{a_f}-K^{a_p}_{.b_sa_f}.
\end{aligned}
\tag{23}
$$

We note that for symmetric linear connections the d–torsion is induced as a pure anholonomic effect.

In a similar manner, putting non–vanishing coefficients (7) into the formula for curvature,

$$
\begin{aligned}
R\left(\delta_{\overline{\tau}},\delta_{\overline{\gamma}}\right)\delta_{\overline{\beta}} &= R_{\overline{\beta}\ \overline{\gamma}\overline{\tau}}^{\ \overline{\alpha}}\delta_{\overline{\alpha}},\\
R_{\overline{\beta}\ \overline{\gamma}\overline{\tau}}^{\ \overline{\alpha}} &= \delta_{\overline{\tau}}\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}-\delta_{\overline{\gamma}}\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\delta}}+\Gamma^{\overline{\varphi}}_{\overline{\beta}\overline{\gamma}}\Gamma^{\overline{\alpha}}_{\overline{\varphi}\overline{\tau}}-\Gamma^{\overline{\varphi}}_{\overline{\beta}\overline{\tau}}\Gamma^{\overline{\alpha}}_{\overline{\varphi}\overline{\gamma}}+\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\varphi}}w^{\overline{\varphi}}_{\overline{\gamma}\overline{\tau}},
\end{aligned}
$$

we can compute the components of d–curvatures

$$
\begin{aligned}
R^{\ i}_{h.jk} &= \delta_{k}L^{i}_{.hj}-\delta_{j}L^{i}_{.hk}+L^{m}_{.hj}L^{i}_{mk}-L^{m}_{.hk}L^{i}_{mj}-C^{i}_{.h\overline{a}}\Omega^{\overline{a}}_{.jk},\\
R^{\ \overline{a}}_{\overline{b}.jk} &= \delta_{k}L^{\overline{a}}_{.\overline{b}j}-\delta_{j}L^{\overline{a}}_{.\overline{b}k}+L^{\overline{c}}_{.\overline{b}j}L^{\overline{a}}_{.\overline{c}k}-L^{\overline{c}}_{.\overline{b}k}L^{\overline{a}}_{.\overline{c}j}-C^{\overline{a}}_{.\overline{b}\overline{c}}\Omega^{\overline{c}}_{.jk},\\
P^{\ i}_{j.k\overline{a}} &= \partial_{k}L^{i}_{.jk}+C^{i}_{.j\overline{b}}T^{\overline{b}}_{.k\overline{a}}-(\partial_{k}C^{i}_{.j\overline{a}}+L^{i}_{.lk}C^{l}_{.j\overline{a}}-L^{l}_{.jk}C^{i}_{.l\overline{a}}-L^{\overline{c}}_{.\overline{a}k}C^{i}_{.j\overline{c}}),\\
P^{\ \overline{c}}_{\overline{b}.k\overline{a}} &= \delta_{\overline{a}}L^{\overline{c}}_{.\overline{b}k}+C^{\overline{c}}_{.\overline{b}\overline{d}}T^{\overline{d}}_{.k\overline{a}}-(\partial_{k}C^{\overline{c}}_{.\overline{b}\overline{a}}+L^{\overline{c}}_{.\overline{d}k}C^{\overline{d}}_{.\overline{b}\overline{a}}-L^{\overline{d}}_{.\overline{b}k}C^{\overline{c}}_{.\overline{d}\overline{a}}-L^{\overline{d}}_{.\overline{a}k}C^{\overline{c}}_{.\overline{b}\overline{d}}),\\
S^{\ i}_{j.\overline{b}\overline{c}} &= \delta_{\overline{c}}C^{i}_{.j\overline{b}}-\delta_{\overline{b}}C^{i}_{.j\overline{c}}+C^{h}_{.j\overline{b}}C^{i}_{.h\overline{c}}-C^{h}_{.j\overline{c}}C^{i}_{h\overline{b}},\\
S^{\ \overline{a}}_{\overline{b}.\overline{c}\overline{d}} &= \delta_{\overline{d}}C^{\overline{a}}_{.\overline{b}\overline{c}}-\delta_{\overline{c}}C^{\overline{a}}_{.\overline{b}\overline{d}}+C^{\overline{e}}_{.\overline{b}\overline{c}}C^{\overline{a}}_{.\overline{e}\overline{d}}-C^{\overline{e}}_{.\overline{b}\overline{d}}C^{\overline{a}}_{.\overline{e}\overline{c}},\\
W^{.a_f}_{b_f.c_fe_f} &= \delta_{e_f}K^{a_f}_{.b_fc_f}-\delta_{c_f}K^{a_f}_{.b_fe_f}+K^{h_f}_{.b_fc_f}K^{a_f}_{h_fe_f}\\
&\quad -K^{h_f}_{.b_fe_f}K^{a_f}_{h_fc_f}-Q^{a_f}_{.b_fa_p}\Omega^{a_p}_{.c_fe_f},
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
W^{.a_p}_{b_s.c_f e_f} &= \delta_{e_f} K^{a_p}_{.b_s c_f} - \delta_{c_f} K^{a_p}_{.b_s e_f} + K^{c_p}_{.b_s c_f} K^{a_p}_{.c_p e_f} \\
&\quad - K^{c_p}_{.b_s e_f} L^{a_p}_{.c_p c_f} - K^{a_p}_{.b_s c_p} \Omega^{c_p}_{.c_f e_f}, \\
Z^{.a_f}_{b_f.c_f e_f} &= \partial_{e_p} K^{a_f}_{.b_f c_f} + Q^{a_f}_{.b_f b_p} T^{b_p}_{.c_f e_p} \\
&\quad -(\partial_{c_f} Q^{a_f}_{.b_f e_p} + K^{a_f}_{.h_f c_f} Q^{h_f}_{.b_f c_p} - K^{h_f}_{.b_f c_f} Q^{a_f}_{.h_f e_p} - K^{c_p}_{.e_p c_f} C^{a_f}_{.b_f c_p}), \\
Z^{.c_s}_{b_r.c_f e_p} &= \delta_{e_p} K^{c_s}_{.b_r c_f} + K^{c_s}_{.b_r d_f} T^{d_f}_{.c_f e_p} \\
&\quad -(\partial_{c_f} C^{c_s}_{.b_r e_p} + K^{c_s}_{.d_f c_f} C^{d_f}_{.b_r e_p} - K^{d_t}_{.b_r c_f} C^{c_s}_{.d_t e_p} - K^{d_t}_{.e_p c_f} C^{c_s}_{.b_r d_t}), \\
Y^{.a_f}_{b_f.c_p e_p} &= \delta_{e_p} Q^{a_f}_{.b_f c_p} - \delta_{c_p} Q^{a_f}_{.b_f e_p} + Q^{d_f}_{.b_f c_p} Q^{a_f}_{.d_f e_p} - Q^{d_f}_{.b_f e_p} Q^{a_f}_{d_f c_p}.
\end{aligned}
$$

where $f < p, s, r, t$.

3.4 Einstein equations with respect to ha–frames

The Ricci tensor

$$R_{\overline{\beta}\overline{\gamma}} = R_{\overline{\beta}\ \overline{\gamma}\overline{\alpha}}^{\ \overline{\alpha}}$$

has the d–components

$$
\begin{aligned}
R_{ij} &= R^{.k}_{i.jk}, \quad R_{i\overline{a}} = -{}^2P_{i\overline{a}} = -P^{.k}_{i.k\overline{a}}, \\
R_{\overline{a}i} &= {}^1P_{\overline{a}i} = P^{.\overline{b}}_{\overline{a}.i\overline{b}}, \quad R_{\overline{a}\overline{b}} = S^{.\overline{c}}_{\overline{a}.\overline{b}\overline{c}} \\
R_{b_f c_f} &= W^{.a_f}_{b_f.c_f a_f}, \quad R_{e_p b_f} = -{}^2P_{b_f e_p} = -Z^{.a_f}_{b_f.a_f e_p}, \\
R_{b_r c_f} &= {}^1P_{b_r c_f} = Z^{.e_s}_{b_r.c_f e_s}.
\end{aligned} \tag{25}
$$

The Ricci d-tensor is non symmetric.

If a higher order d-metric of type (21) is defined in $V^{(\overline{n})}$, we can compute the scalar curvature

$$\overline{R} = g^{\overline{\beta}\overline{\gamma}} R_{\overline{\beta}\overline{\gamma}}.$$

of a d-connection D,

$$\overline{R} = \widehat{R} + \overline{S}, \tag{26}$$

where $\widehat{R} = g^{ij} R_{ij}$ and $\overline{S} = h^{\overline{a}\overline{b}} S_{\overline{a}\overline{b}}$.

The h-v parametrization of the gravitational field equations in ha–spacetimes is obtained by introducing the values (25) and (26) into the Einstein's equations

$$R_{\overline{\beta}\overline{\gamma}} - \frac{1}{2} g_{\overline{\beta}\overline{\gamma}} \overline{R} = k \Upsilon_{\overline{\beta}\overline{\gamma}},$$

and written

$$
\begin{aligned}
R_{ij} - \frac{1}{2}\left(\widehat{R} + \overline{S}\right) g_{ij} &= k\Upsilon_{ij}, \\
S_{\overline{a}\overline{b}} - \frac{1}{2}\left(\widehat{R} + \overline{S}\right) h_{\overline{a}\overline{b}} &= k\Upsilon_{\overline{a}\overline{b}}, \\
{}^1P_{\overline{a}i} = k\Upsilon_{\overline{a}i}, \quad {}^1P_{a_p b_f} &= k\Upsilon_{a_p b_f} \\
{}^2P_{i\overline{a}} = -k\Upsilon_{i\overline{a}}, \quad {}^2P_{a_s b_f} &= -k\Upsilon_{a_f b_p},
\end{aligned} \tag{27}
$$

where $\Upsilon_{ij}, \Upsilon_{\overline{a}\overline{b}}, \Upsilon_{\overline{a}i}, \Upsilon_{i\overline{a}}, \Upsilon_{a_p b_f}, \Upsilon_{a_f b_p}$ are the h-v–components of the energy–momentum d–tensor field $\Upsilon_{\overline{\beta}\overline{\gamma}}$ (which includes possible cosmological constants, contributions of anholonomy d–torsions (23) and matter) and k is the coupling constant.

We note that, in general, the ha–torsions are not vanishing. Nevetheless, for a (pseudo)–Riemannian spacetime with induced anholonomic anisotropies it is not necessary to consider an additional to (27) system of equations for torsion becouse in this case the torsion structure is an anholonomic effect wich becames trivial with respect to holonomic frames of reference.

If a ha–spacetime structure is associated to a generic nonzero torsion, we could consider additionally, for instance, as in [33], a system of algebraic d–field equations with a source $S^{\overline{\alpha}}_{\ \overline{\beta}\overline{\gamma}}$ for a locally anisotropic spin density of matter (if we construct a variant of higher order anisotropic Einstein–Cartan theory):

$$T^{\overline{\gamma}}_{\ \overline{\alpha}\overline{\beta}} + 2\delta^{\overline{\gamma}}_{\ [\overline{\alpha}} T^{\overline{\delta}}_{\ \overline{\beta}]\overline{\delta}} = \kappa S^{\overline{\gamma}}_{\ \overline{\alpha}\overline{\beta}.}$$

In a more general case we have to introduce some new constraints and/or dynamical equations for torsions and nonlinear connections which are induced from (super) string theory and/ or higher order anisotropic supergravity [27, 28]. Two variants of gauge dynamical field equations with both frame like and torsion variables will be considered in the Section 5 and 6 of this paper.

4 Gauge Fields on Ha–Spaces

This section is devoted to gauge field theories on spacetimes provided with higher order anisotropic anholonomic frame structures.

4.1 Bundles on ha–spaces

Let us consider a principal bundle $\left(\mathcal{P}, \pi, Gr, V^{(\overline{n})}\right)$ over a ha–spacetime $V^{(\overline{n})}$ ($\mathcal{P}$ and $V^{(\overline{n})}$ are called respectively the base and total spaces) with the structural group Gr and surjective map $\pi : \mathcal{P} \rightarrow V^{(\overline{n})}$ (on geometry of bundle spaces see, for instance, [4, 17, 23]). At every point $u = (x, y_{(1)}, \dots, y_{(z)}) \in V^{(\overline{n})}$ there is a vicinity $\mathcal{U} \subset V^{(\overline{n})}, u \in \mathcal{U}$, with trivializing $\mathcal{P}$ diffeomorphisms f and $\varphi :$

$$\begin{aligned} f_{\mathcal{U}} : \ \pi^{-1}(\mathcal{U}) &\rightarrow \mathcal{U}\times Gr, \qquad f(p) = (\pi(p), \varphi(p)), \\ \varphi_{\mathcal{U}} : \ \pi^{-1}(\mathcal{U}) &\rightarrow Gr, \varphi(pq) = \varphi(p)\, q \end{aligned}$$

for every group element $q \in Gr$ and point $p \in \mathcal{P}$. We remark that in the general case for two open regions

$$\mathcal{U}, \mathcal{V} \subset V^{(\overline{n})}, \mathcal{U}\cap\mathcal{V} \neq \emptyset, f_{\mathcal{U}|_p} \neq f_{\mathcal{V}|_p}, \text{ even } p \in \mathcal{U}\cap\mathcal{V}.$$

Transition functions $g_{\mathcal{U}\mathcal{V}}$ are defined

$$g_{\mathcal{U}\mathcal{V}} : \mathcal{U}\cap\mathcal{V} \rightarrow Gr, g_{\mathcal{U}\mathcal{V}}(u) = \varphi_{\mathcal{U}}(p)\left(\varphi_{\mathcal{V}}(p)^{-1}\right), \pi(p) = u.$$

Hereafter we shall omit, for simplicity, the specification of trivializing regions of maps and denote, for example, $f \equiv f_{\mathcal{U}}, \varphi \equiv \varphi_{\mathcal{U}},\ s \equiv s_{\mathcal{U}}$, if this will not give rise to ambiguities.

Let θ be the canonical left invariant 1-form on Gr with values in algebra Lie $\mathcal{G}$ of group Gr uniquely defined from the relation $\theta(q) = q$, for every $q \in \mathcal{G}$, and consider a 1-form ω on $\mathcal{U} \subset V^{(\overline{n})}$ with values in $\mathcal{G}$. Using θ and ω, we can locally define the connection form Θ in $\mathcal{P}$ as a 1-form:

$$\Theta = \varphi^* \theta + Ad\ \varphi^{-1} (\pi^* \omega) \tag{28}$$

where $\varphi^*\theta$ and $\pi^*\omega$ are, respectively, 1–forms induced on $\pi^{-1}(\mathcal{U})$ and $\mathcal{P}$ by maps φ and π and $\omega = s^*\Theta$. The adjoint action on a form λ with values in $\mathcal{G}$ is defined as

$$\left(Ad\ \varphi^{-1}\lambda\right)_p = \left(Ad\ \varphi^{-1}(p)\right)\lambda_p$$

where λ_p is the value of form λ at point $p \in \mathcal{P}$.

Introducing a basis $\{\Delta_{\widehat{a}}\}$ in $\mathcal{G}$ (index $\widehat{a}$ enumerates the generators making up this basis), we write the 1-form ω on $V^{(\overline{n})}$ as

$$\omega = \Delta_{\widehat{a}} \omega^{\widehat{a}}(u)\,,\ \omega^{\widehat{a}}(u) = \omega^{\widehat{a}}_{\overline{\mu}}(u)\,\delta u^{\overline{\mu}} \tag{29}$$

where $\delta u^{\overline{\mu}} = \left(dx^i, \delta y^{\overline{a}}\right)$ and the Einstein summation rule on indices $\widehat{a}$ and $\overline{\mu}$ is used. Functions $\omega^{\widehat{a}}_{\overline{\mu}}(u)$ from (29) are called the components of Yang-Mills fields on ha-spacetime $V^{(\overline{n})}$. Gauge transforms of ω can be interpreted as transition relations for $\omega_{\mathcal{U}}$ and $\omega_{\mathcal{V}}$, when $u \in \mathcal{U} \cap \mathcal{V}$,

$$(\omega_{\mathcal{U}})_u = (g^*_{\mathcal{UV}}\theta)_u + Ad\ g_{\mathcal{UV}}(u)^{-1}(\omega_{\mathcal{V}})_u\,. \tag{30}$$

To relate $\omega^{\widehat{a}}_{\overline{\mu}}$ with a covariant derivation we shall consider a vector bundle Ξ associated to $\mathcal{P}$. Let $\rho : Gr \to GL(\mathcal{R}^s)$ and $\rho' : \mathcal{G} \to End(E^s)$ be, respectively, linear representations of group Gr and Lie algebra $\mathcal{G}$ (where $\mathcal{R}$ is the real number field). Map ρ defines a left action on Gr and associated vector bundle $\Xi = P \times \mathcal{R}^s/Gr,\ \pi_E : E \to V^{(\overline{n})}$. Introducing the standard basis $\xi_{\underline{i}} = \{\xi_{\underline{1}}, \xi_{\underline{2}}, ..., \xi_{\underline{s}}\}$ in $\mathcal{R}^s$, we can define the right action on $P\times\ \mathcal{R}^s$, $\left((p,\xi)\,q = \left(pq, \rho\left(q^{-1}\right)\xi\right), q \in Gr\right)$, the map induced from $\mathcal{P}$

$$p : \mathcal{R}^s \to \pi_E^{-1}(u)\,, \quad (p(\xi) = (p\xi)\,Gr, \xi \in \mathcal{R}^s, \pi(p) = u)$$

and a basis of local sections $e_{\underline{i}} : U \to \pi_E^{-1}(U)\,,\ e_{\underline{i}}(u) = s(u)\,\xi_{\underline{i}}$. Every section $\varsigma : V^{(\overline{n})} \to \Xi$ can be written locally as $\varsigma = \varsigma^i e_i, \varsigma^i \in C^\infty(\mathcal{U})$. To every vector field X on $V^{(\overline{n})}$ and Yang-Mills field $\omega^{\widehat{a}}$ on $\mathcal{P}$ we associate operators of covariant derivations:

$$\nabla_X \zeta = e_{\underline{i}}\left[X\zeta^{\underline{i}} + B(X)^{\underline{i}}_{\underline{j}}\,\zeta^{\underline{j}}\right],\ B(X) = (\rho' X)_{\widehat{a}}\,\omega^{\widehat{a}}(X)\,. \tag{31}$$

The transform (30) and operators (31) are interrelated by these transition transforms for values $e_{\underline{i}}, \zeta^{\underline{i}}$, and $B_{\overline{\mu}}$:

$$\begin{aligned} e^{\mathcal{V}}_{\underline{i}}(u) &= [\rho g_{\mathcal{UV}}(u)]^{\underline{j}}_{\underline{i}}\, e^{\mathcal{U}}_{\underline{i}},\ \zeta^{\underline{i}}_{\mathcal{U}}(u) = [\rho g_{\mathcal{UV}}(u)]^{\underline{j}}_{\underline{i}}\,\zeta^{\underline{i}}_{\mathcal{V}}, \\ B^{\mathcal{V}}_{\overline{\mu}}(u) &= [\rho g_{\mathcal{UV}}(u)]^{-1}\,\delta_{\overline{\mu}}\,[\rho g_{\mathcal{UV}}(u)] + [\rho g_{\mathcal{UV}}(u)]^{-1}\,B^{\mathcal{U}}_{\overline{\mu}}(u)\,[\rho g_{\mathcal{UV}}(u)]\,, \end{aligned} \tag{32}$$

where $B^{\mathcal{U}}_{\overline{\mu}}(u) = B^{\overline{\mu}}\left(\delta/du^{\overline{\mu}}\right)(u)$.

Using (32), we can verify that the operator $\nabla^{\mathcal{U}}_X$, acting on sections of $\pi_\Xi : \Xi \to V^{(\overline{n})}$ according to definition (31), satisfies the properties

$$\begin{aligned} \nabla^{\mathcal{U}}_{f_1X+f_2Y} &= f_1\nabla^{\mathcal{U}}_X + f_2\nabla^{\mathcal{U}}_X,\ \nabla^{\mathcal{U}}_X(f\zeta) = f\nabla^{\mathcal{U}}_X\zeta + (Xf)\zeta, \\ \nabla^{\mathcal{U}}_X\zeta &= \nabla^{\mathcal{V}}_X\zeta, \quad u \in \mathcal{U}\cap\mathcal{V}, f_1, f_2 \in C^\infty(\mathcal{U}). \end{aligned}$$

So, we can conclude that the Yang–Mills connection in the vector bundle $\pi_\Xi : \Xi \to V^{(\overline{n})}$ is not a general one, but is induced from the principal bundle $\pi : \mathcal{P} \to V^{(\overline{n})}$ with structural group Gr.

The curvature $\mathcal{K}$ of connection Θ from (28) is defined as

$$\mathcal{K} = D\Theta,\ D = \widehat{H}\circ d \tag{33}$$

where d is the operator of exterior derivation acting on $\mathcal{G}$-valued forms as

$$d\left(\Delta_{\widehat{a}}\otimes\chi^{\widehat{a}}\right) = \Delta_{\widehat{a}}\otimes d\chi^{\widehat{a}}$$

and $\widehat{H}$ is the horizontal projecting operator acting, for example, on the 1-form λ as $\left(\widehat{H}\lambda\right)_P(X_p) = \lambda_p(H_pX_p)$, where H_p projects on the horizontal subspace

$$\mathcal{H}_p \in P_p\left[X_p \in \mathcal{H}_p \text{ is equivalent to } \Theta_p(X_p) = 0\right].$$

We can express (33) locally as

$$\mathcal{K} = Ad\ \varphi^{-1}_{\mathcal{U}}(\pi^*\mathcal{K}_{\mathcal{U}}) \tag{34}$$

where

$$\mathcal{K}_{\mathcal{U}} = d\omega_{\mathcal{U}} + \frac{1}{2}\left[\omega_{\mathcal{U}}, \omega_{\mathcal{U}}\right]. \tag{35}$$

The exterior product of $\mathcal{G}$-valued form (35) is defined as

$$\left[\Delta_{\widehat{a}}\otimes\lambda^{\widehat{a}}, \Delta_{\widehat{b}}\otimes\xi^{\widehat{b}}\right] = \left[\Delta_{\widehat{a}}, \Delta_{\widehat{b}}\right]\otimes\lambda^{\widehat{a}}\bigwedge\xi^{\widehat{b}},$$

where the antisymmetric tensorial product is denoted $\lambda^{\widehat{a}}\bigwedge\xi^{\widehat{b}} = \lambda^{\widehat{a}}\xi^{\widehat{b}} - \xi^{\widehat{b}}\lambda^{\widehat{a}}$.

Introducing structural coefficients $f_{\widehat{b}\widehat{c}}{}^{\widehat{a}}$ of $\mathcal{G}$ satisfying

$$\left[\Delta_{\widehat{b}}, \Delta_{\widehat{c}}\right] = f_{\widehat{b}\widehat{c}}{}^{\widehat{a}}\Delta_{\widehat{a}}$$

we can rewrite (35) in a form more convenient for local considerations:

$$\mathcal{K}_{\mathcal{U}} = \Delta_{\widehat{a}}\otimes\mathcal{K}^{\widehat{a}}_{\overline{\mu}\overline{\nu}}\delta u^{\overline{\mu}}\bigwedge\delta u^{\overline{\nu}} \tag{36}$$

where

$$\mathcal{K}^{\widehat{a}}_{\overline{\mu}\overline{\nu}} = \frac{\delta\omega^{\widehat{a}}_{\overline{\nu}}}{\partial u^{\overline{\mu}}} - \frac{\delta\omega^{\widehat{a}}_{\overline{\mu}}}{\partial u^{\overline{\nu}}} + \frac{1}{2}f_{\widehat{b}\widehat{c}}{}^{\widehat{a}}\left(\omega^{\widehat{b}}_{\overline{\mu}}\omega^{\widehat{c}}_{\overline{\nu}} - \omega^{\widehat{b}}_{\overline{\nu}}\omega^{\widehat{c}}_{\overline{\mu}}\right)..$$

This subsection ends by considering the problem of reduction of the local anisotropic gauge symmetries and gauge fields to isotropic ones. For local trivial considerations we can consider that with respect to holonomic frames the higher order anisotropic Yang-Mills fields reduce to usual ones on (pseudo) Riemannian spaces.

4.2 Yang-Mills equations on ha-spaces

Interior gauge symmetries are associated to semisimple structural groups. On the principal bundle $\left(\mathcal{P}, \pi, Gr, V^{(\overline{n})}\right)$ with nondegenerate Killing form for semisimple group Gr we can define the generalized bundle metric

$$h_p\left(X_p, Y_p\right) = G_{\pi(p)}\left(d\pi_P X_P, d\pi_P Y_P\right) + K\left(\Theta_P\left(X_P\right), \Theta_P\left(X_P\right)\right), \qquad (37)$$

where $d\pi_P$ is the differential of map $\pi : \mathcal{P} \to V^{(\overline{n})}$, $G_{\pi(p)}$ is locally generated as the ha-metric (21), and K is the Killing form on $\mathcal{G}$:

$$K\left(\Delta_{\widehat{a}}, \Delta_{\widehat{b}}\right) = f_{\widehat{bd}}{}^{\widehat{c}} f_{\widehat{ac}}{}^{\widehat{d}} = K_{\widehat{ab}}.$$

Using the metric $g_{\overline{\alpha\beta}}$ on $V^{(\overline{n})}$ (respectively, $h_P\left(X_P, Y_P\right)$ on $\mathcal{P}$) we can introduce operators $*_G$ and $\widehat{\delta}_G$ acting in the space of forms on $V^{(\overline{n})}$ ($*_H$ and $\widehat{\delta}_H$ acting on forms on $\mathcal{P}$)). Let $e_{\overline{\mu}}$ be an orthonormalized frame on $\mathcal{U} \subset V^{(\overline{n})}$, locally adapted to the N–connection structure, i. .e. being related via some local distinguisherd linear transforms to a ha–frame (16) and $e^{\overline{\mu}}$ be the adjoint coframe. Locally

$$G = \sum_{\overline{\mu}} \eta\left(\overline{\mu}\right) e^{\overline{\mu}} \otimes e^{\overline{\mu}},$$

where $\eta_{\overline{\mu\mu}} = \eta\left(\overline{\mu}\right) = \pm 1$, $\overline{\mu} = 1, 2, ..., \overline{n}$, and the Hodge operator $*_G$ can be defined as $*_G : \Lambda'\left(V^{(\overline{n})}\right) \to \Lambda^{\overline{n}}\left(V^{(\overline{n})}\right)$, or, in explicit form, as

$$*_G\left(e^{\overline{\mu}_1} \bigwedge ... \bigwedge e^{\overline{\mu}_r}\right) = \eta\left(\overline{\nu}_1\right) ... \eta\left(\overline{\nu}_{\overline{n}-r}\right) \times \qquad (38)$$

$$sign\begin{pmatrix} 1 & 2 & \dots & r & r+1 & \dots & \overline{n} \\ \overline{\mu}_1 & \overline{\mu}_2 & \dots & \overline{\mu}_r & \overline{\nu}_1 & \dots & \overline{\nu}_{\overline{n}-r} \end{pmatrix} e^{\overline{\nu}_1} \bigwedge ... \bigwedge e^{\overline{\nu}_{\overline{n}-r}}.$$

Next, we define the operator

$$*_G^{-1} = \eta\left(1\right) ... \eta\left(\overline{n}\right) \left(-1\right)^{r(\overline{n}-r)} *_G$$

and introduce the scalar product on forms $\beta_1, \beta_2, ... \subset \Lambda^r\left(V^{(\overline{n})}\right)$ with compact carrier:

$$\left(\beta_1, \beta_2\right) = \eta\left(1\right) ... \eta\left(n_E\right) \int \beta_1 \bigwedge *_G \beta_2.$$

The operator $\widehat{\delta}_G$ is defined as the adjoint to d associated to the scalar product for forms, specified for r-forms as

$$\widehat{\delta}_G = \left(-1\right)^r *_G^{-1} \circ d \circ *_G. \qquad (39)$$

We remark that operators $*_H$ and δ_H acting in the total space of $\mathcal{P}$ can be defined similarly to (38) and (39), but by using metric (37). Both these operators also act in the space of $\mathcal{G}$-valued forms:

$$*\left(\Delta_{\widehat{a}} \otimes \varphi^{\widehat{a}}\right) = \Delta_{\widehat{a}} \otimes (*\varphi^{\widehat{a}}),$$

$$\widehat{\delta}\left(\Delta_{\widehat{a}} \otimes \varphi^{\widehat{a}}\right) = \Delta_{\widehat{a}} \otimes (\widehat{\delta}\varphi^{\widehat{a}}).$$

The form λ on $\mathcal{P}$ with values in $\mathcal{G}$ is called horizontal if $\widehat{H}\lambda = \lambda$ and equivariant if $R^*(q)\lambda = Ad\ q^{-1}\varphi,\ \forall g \in Gr, R(q)$ being the right shift on $\mathcal{P}$. We can verify that equivariant and horizontal forms also satisfy the conditions

$$\lambda = Ad\ \varphi_{\mathcal{U}}^{-1}(\pi^*\lambda), \qquad \lambda_{\mathcal{U}} = S_{\mathcal{U}}^*\lambda,$$

$$(\lambda_{\mathcal{V}})_{\mathcal{U}} = Ad\ \left(g_{\mathcal{U}\mathcal{V}}(u)\right)^{-1}(\lambda_{\mathcal{U}})_u.$$

Now, we can define the field equations for curvature (34) and connection (28):

$$\Delta\mathcal{K} = 0, \tag{40}$$

$$\nabla\mathcal{K} = 0, \tag{41}$$

where $\Delta = \widehat{H} \circ \widehat{\delta}_H$. Equations (40) are similar to the well-known Maxwell equations and for non-Abelian gauge fields are called Yang-Mills equations. The structural equations (41) are called the Bianchi identities.

The field equations (40) do not have a physical meaning because they are written in the total space of the bundle Ξ and not on the base anisotropic spacetime $V^{(\overline{n})}$. But this difficulty may be obviated by projecting the mentioned equations on the base. The 1-form $\Delta\mathcal{K}$ is horizontal by definition and its equivariance follows from the right invariance of metric (37). So, there is a unique form $(\Delta\mathcal{K})_{\mathcal{U}}$ satisfying

$$\Delta\mathcal{K} = Ad\ \varphi_{\mathcal{U}}^{-1}\pi^*(\Delta\mathcal{K})_{\mathcal{U}}.$$

The projection of (40) on the base can be written as $(\Delta\mathcal{K})_{\mathcal{U}} = 0$. To calculate $(\Delta\mathcal{K})_{\mathcal{U}}$, we use the equality [4, 24]

$$d\left(Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right) = Ad\ \ \varphi_{\mathcal{U}}^{-1}\ d\lambda - \left[\varphi_{\mathcal{U}}^*\theta, Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right]$$

where λ is a form on $\mathcal{P}$ with values in $\mathcal{G}$. For r-forms we have

$$\widehat{\delta}\left(Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right) = Ad\ \varphi_{\mathcal{U}}^{-1}\widehat{\delta}\lambda - (-1)^r *_H \{\left[\varphi_{\mathcal{U}}^*\theta, *_H Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right]$$

and, as a consequence,

$$\widehat{\delta}\mathcal{K} = Ad\ \varphi_{\mathcal{U}}^{-1}\{\widehat{\delta}_H\pi^*\mathcal{K}_{\mathcal{U}} + *_H^{-1}[\pi^*\omega_{\mathcal{U}}, *_H\pi^*\mathcal{K}_{\mathcal{U}}]\} - *_H^{-1}\left[\Theta, Ad\ \varphi_{\mathcal{U}}^{-1} *_H(\pi^*\mathcal{K})\right]. \tag{42}$$

By using straightforward calculations in the adapted dual basis on $\pi^{-1}(\mathcal{U})$ we can verify the equalities

$$\left[\Theta, Ad\ \varphi_{\mathcal{U}}^{-1}\ \ *_H(\pi^*\mathcal{K}_{\mathcal{U}})\right] = 0, \widehat{H}\delta_H(\pi^*\mathcal{K}_{\mathcal{U}}) = \pi^*\left(\widehat{\delta}_G\mathcal{K}\right), \tag{43}$$

$$*_H^{-1}[\pi^*\omega_{\mathcal{U}}, *_H(\pi^*\mathcal{K}_{\mathcal{U}})] = \pi^*\{*_G^{-1}[\omega_{\mathcal{U}}, *_G\mathcal{K}_{\mathcal{U}}]\}.$$

¿From (42) and (43) one follows that

$$(\Delta\mathcal{K})_{\mathcal{U}} = \widehat{\delta}_G\mathcal{K}_{\mathcal{U}} + *_G^{-1}[\omega_{\mathcal{U}}, *_G\mathcal{K}_{\mathcal{U}}]. \tag{44}$$

Taking into account (44) and (39), we prove that projection on the base of equations (40) and (41) can be expressed respectively as

$$*_G^{-1} \circ d \circ *_G\mathcal{K}_{\mathcal{U}} + *_G^{-1}[\omega_{\mathcal{U}}, *_G\mathcal{K}_{\mathcal{U}}] = 0. \tag{45}$$

$$d\mathcal{K}_{\mathcal{U}} + [\omega_{\mathcal{U}}, \mathcal{K}_{\mathcal{U}}] = 0.$$

Equations (45) (see (44)) are gauge–invariant because

$$(\Delta\mathcal{K})_{\mathcal{U}} = Ad\ g_{\mathcal{U}\mathcal{V}}^{-1}\,(\Delta\mathcal{K})_{\mathcal{V}}\,.$$

By using formulas (36)-(39) we can rewrite (45) in coordinate form

$$D_{\overline{\nu}}\left(G^{\overline{\nu}\overline{\lambda}}\mathcal{K}^{\widehat{a}}_{\ \overline{\lambda}\overline{\mu}}\right) + f_{\widehat{b}\widehat{c}}^{\ \ \widehat{a}} g^{\overline{\nu}\overline{\lambda}}\omega^{\widehat{b}}_{\overline{\lambda}}\mathcal{K}^{\widehat{c}}_{\ \overline{\nu}\overline{\mu}} = 0, \tag{46}$$

where $D_{\overline{\nu}}$ is a compatible with metric covariant derivation on ha-spacetime (46).

We point out that for our bundles with semisimple structural groups the Yang-Mills equations (40) (and, as a consequence, their horizontal projections (45), or (46)) can be obtained by variation of the action

$$I = \int \mathcal{K}^{\widehat{a}}_{\ \overline{\mu}\overline{\nu}}\mathcal{K}^{\widehat{b}}_{\ \overline{\alpha}\overline{\beta}}G^{\overline{\mu}\overline{\alpha}}g^{\overline{\nu}\overline{\beta}}K_{\widehat{a}\widehat{b}}\left|g_{\overline{\alpha}\overline{\beta}}\right|^{1/2} dx^1...dx^n\delta y^1_{(1)}...\delta y^{m_1}_{(1)}...\delta y^1_{(z)}...\delta y^{m_z}_{(z)}. \tag{47}$$

Equations for extremals of (47) have the form

$$K_{\widehat{r}\widehat{b}}g^{\overline{\lambda}\overline{\alpha}}g^{\overline{\kappa}\overline{\beta}}D_{\overline{\alpha}}\mathcal{K}^{\widehat{b}}_{\ \overline{\lambda}\overline{\beta}} - K_{\widehat{a}\widehat{b}}g^{\overline{\kappa}\overline{\alpha}}g^{\overline{\nu}\overline{\beta}}f_{\widehat{r}\widehat{l}}^{\ \ \widehat{a}}\omega^{\widehat{l}}_{\overline{\nu}}\mathcal{K}^{\widehat{b}}_{\ \overline{\alpha}\overline{\beta}} = 0,$$

which are equivalent to "pure" geometric equations (46) (or (45)) due to nondegeneration of the Killing form $K_{\widehat{r}\widehat{b}}$ for semisimple groups.

To take into account gauge interactions with matter fields (sections of vector bundle Ξ on $V^{(\overline{n})}$) we have to introduce a source 1–form $\mathcal{J}$ in equations (40) and to write them

$$\Delta\mathcal{K} = \mathcal{J} \tag{48}$$

Explicit constructions of $\mathcal{J}$ require concrete definitions of the bundle Ξ; for example, for spinor fields an invariant formulation of the Dirac equations on ha–spaces is necessary. We omit spinor considerations in this paper (see [26, 29]).

5 Gauge Ha-gravity

A considerable body of work on the formulation of gauge gravitational models on isotropic spaces is based on application of nonsemisimple groups, for example, of Poincare and affine groups, as structural gauge groups (see critical analysis and original results in [6, 33, 14, 8, 35, 34, 22]). The main impediment to developing such models is caused by the degeneration of Killing forms for nonsemisimple groups, which make it impossible to construct consistent variational gauge field theories (functional (47) and extremal equations are degenerate in these cases). There are at least two possibilities to get around the mentioned difficulty. The first is to realize a minimal extension of the nonsemisimple group to a semisimple one, similar to the extension of the Poincare group to the de Sitter group considered in [23, 24, 34]. The second possibility is to introduce into consideration the bundle of adapted affine frames on la-space $V^{(\overline{n})}$, to use an auxiliary nondegenerate bilinear form $a_{\widehat{a}\widehat{b}}$ instead of the degenerate Killing form $K_{\widehat{a}\widehat{b}}$ and to consider a "pure" geometric method, illustrated in the previous section, of definition of gauge field equations. Projecting on the base $V^{(\overline{n})}$, we shall obtain gauge

gravitational field equations on ha-space having a form similar to Yang-Mills equations.

The goal of this section is to prove that a specific parametrization of components of the Cartan connection in the bundle of adapted affine frames on $V^{(\overline{n})}$ establishes an equivalence between Yang-Mills equations (48) and Einstein equations (27) on ha–spaces.

5.1 Bundles of linear ha–frames

Let $(X_{\overline{\alpha}})_u = (X_i, X_{\overline{a}})_u = (X_i, X_{a_1}, ..., X_{a_z})_u$ be a frame locally adapted to the N–conection structure at a point $u \in V^{(\overline{n})}$. We consider a local right distinguished action of matrices

$$A_{\overline{\alpha}'}{}^{\overline{\alpha}} = \begin{pmatrix} A_{i'}{}^{i} & 0 & ... & 0 \\ 0 & B_{a_1'}{}^{a_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & B_{a_z'}{}^{a_z} \end{pmatrix} \subset GL_{\overline{n}} =$$

$$GL\left(n,\mathcal{R}\right) \oplus GL\left(m_1,\mathcal{R}\right) \oplus ... \oplus GL\left(m_z,\mathcal{R}\right).$$

Nondegenerate matrices $A_{i'}{}^{i}$ and $B_{j'}{}^{j}$, respectively, transform linearly $X_{i|u}$ into $X_{i'|u} = A_{i'}{}^{i} X_{i|u}$ and $X_{a'_p|u}$ into $X_{a'_p|u} = B_{a'_p}{}^{a_p} X_{a_p|u}$, where $X_{\overline{\alpha}'|u} = A_{\overline{\alpha}'}{}^{\overline{\alpha}} X_{\overline{\alpha}}$ is also an adapted frame at the same point $u \in V^{(\overline{n})}$. We denote by $La\left(V^{(\overline{n})}\right)$ the set of all adapted frames $X_{\overline{\alpha}}$ at all points of $V^{(\overline{n})}$ and consider the surjective map π from $La\left(V^{(\overline{n})}\right)$ to $V^{(\overline{n})}$ transforming every adapted frame $X_{\overline{\alpha}|u}$ and point u into the point u. Every $X_{\overline{\alpha}'|u}$ has a unique representation as $X_{\overline{\alpha}'} = A_{\overline{\alpha}'}{}^{\overline{\alpha}} X^{(0)}_{\overline{\alpha}}$, where $X^{(0)}_{\overline{\alpha}}$ is a fixed distinguished basis in tangent space $T\left(V^{(\overline{n})}\right)$. It is obvious that $\pi^{-1}\left(\mathcal{U}\right), \mathcal{U} \subset V^{(\overline{n})}$, is bijective to $\mathcal{U} \times GL_{\overline{n}}\left(\mathcal{R}\right)$. We can transform $La\left(V^{(\overline{n})}\right)$ in a differentiable manifold taking $\left(u^{\overline{\beta}}, A_{\overline{\alpha}'}{}^{\overline{\alpha}}\right)$ as a local coordinate system on $\pi^{-1}\left(\mathcal{U}\right)$. Now, it is easy to verify that

$$\mathcal{L}a(V^{(\overline{n})}) = (La(V^{(\overline{n})}), V^{(\overline{n})}, GL_{\overline{n}}(\mathcal{R}))$$

is a principal bundle. We call $\mathcal{L}a(V^{(\overline{n})})$ the bundle of linear adapted frames on $V^{(\overline{n})}$.

The next step is to identify the components of, for simplicity, compatible d-connection $\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}$ on $V^{(\overline{n})}$, with the connection in $\mathcal{L}a(V^{(\overline{n})})$

$$\Theta^{\widehat{a}}_{\mathcal{U}} = \omega^{\widehat{a}} = \{\omega^{\widehat{\alpha}\widehat{\beta}}{}_{\overline{\lambda}} \doteq \Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}\}. \tag{49}$$

Introducing (49) in (44), we calculate the local 1-form

$$\left(\Delta\mathcal{R}^{(\Gamma)}\right)_{\mathcal{U}} = \Delta_{\widehat{\alpha\alpha_1}} \otimes (g^{\overline{\nu}\overline{\lambda}} D_{\overline{\lambda}} \mathcal{R}^{\widehat{\alpha\gamma}}{}_{\overline{\nu}\overline{\mu}} + f^{\widehat{\alpha\gamma}}{}_{\widehat{\beta\delta}\widehat{\gamma\varepsilon}} g^{\overline{\nu}\overline{\lambda}} \omega^{\widehat{\beta\delta}}{}_{\lambda} \mathcal{R}^{\widehat{\gamma\varepsilon}}{}_{\overline{\nu}\overline{\mu}}) \delta u^{\overline{\mu}}, \tag{50}$$

where

$$\Delta_{\widehat{\alpha}\widehat{\beta}} = \begin{pmatrix} \Delta_{\widehat{i}\widehat{j}} & 0 & ... & 0 \\ 0 & \Delta_{\widehat{a}_1\widehat{b}_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \Delta_{\widehat{a}_z\widehat{b}_z} \end{pmatrix}$$

is the standard distinguished basis in the Lie algebra of matrices $\mathcal{G}l_{\overline{n}}(\mathcal{R})$ with $(\Delta_{\widehat{i}\widehat{k}})_{jl} = \delta_{ij}\delta_{kl}$ and $\left(\Delta_{\widehat{a}_p\widehat{c}_p}\right)_{b_pd_p} = \delta_{a_pb_p}\delta_{c_pd_p}$ defining the standard bases in $\mathcal{G}l\left(\mathcal{R}^{\overline{n}}\right)$. We have denoted the curvature of connection (49), considered in (50), as

$$\mathcal{R}_{\mathcal{U}}^{(\Gamma)} = \Delta_{\widehat{\alpha}\widehat{\alpha}_1} \otimes \mathcal{R}^{\widehat{\alpha}\widehat{\alpha}_1}{}_{\overline{\nu}\overline{\mu}} X^{\overline{\nu}} \bigwedge X^{\overline{\mu}},$$

where $\mathcal{R}^{\widehat{\alpha}\widehat{\alpha}_1}{}_{\overline{\nu}\overline{\mu}} = R_{\overline{\alpha}_1}{}^{\overline{\alpha}}{}_{\overline{\nu}\overline{\mu}}$ (see curvatures (24)).

5.2 Bundles of affine ha–frames and Einstein equations

Besides the bundles $\mathcal{L}a\left(V^{(\overline{n})}\right)$ on ha-spacetime $V^{(\overline{n})}$, there is another bundle, the bundle of adapted affine frames with structural group $Af_{n_E}(\mathcal{R}) = GL_{n_E}\left(V^{(\overline{n})}\right)$ $\otimes\mathcal{R}^{\overline{n}}$, which can be naturally related to the gravity models on (pseudo) Riemannian spaces. Because as a linear space the Lie Algebra $af_{\overline{n}}(\mathcal{R})$ is a direct sum of $\mathcal{G}l_{\overline{n}}(\mathcal{R})$ and $\mathcal{R}^{\overline{n}}$, we can write forms on $\mathcal{A}a\left(V^{(\overline{n})}\right)$ as $\Phi = (\Phi_1, \Phi_2)$, where Φ_1 is the $\mathcal{G}l_{\overline{n}}(\mathcal{R})$ component and Φ_2 is the $\mathcal{R}^{\overline{n}}$ component of the form Φ. The connection (49), Θ in $\mathcal{L}a\left(V^{(\overline{n})}\right)$, induces the Cartan connection $\overline{\Theta}$ in $\mathcal{A}a\left(V^{(\overline{n})}\right)$; see the isotropic case in [23, 24, 4]. There is only one connection on $\mathcal{A}a\left(V^{(\overline{n})}\right)$ represented as $i^*\overline{\Theta} = (\Theta, \chi)$, where χ is the shifting form and $i: \mathcal{A}a \to \mathcal{L}a$ is the trivial reduction of bundles. If $s_{\mathcal{U}}^{(a)}$ is a local adapted frame in $\mathcal{L}a\left(V^{(\overline{n})}\right)$, then $\overline{s}_{\mathcal{U}}^{(0)} = i \circ s_{\mathcal{U}}$ is a local section in $\mathcal{A}a\left(V^{(\overline{n})}\right)$ and

$$\left(\overline{\Theta}_{\mathcal{U}}\right) = s_{\mathcal{U}}\Theta = (\Theta_{\mathcal{U}}, \chi_{\mathcal{U}}), \tag{51}$$

where $\chi = e_{\widehat{\alpha}} \otimes \chi^{\widehat{\alpha}}_{\overline{\mu}} X^{\overline{\mu}}$, $g_{\overline{\alpha}\overline{\beta}} = \chi^{\widehat{\alpha}}_{\overline{\alpha}} \chi^{\widehat{\beta}}_{\overline{\beta}} \eta_{\widehat{\alpha}\widehat{\beta}}$ ($\eta_{\widehat{\alpha}\widehat{\beta}}$ is diagonal with $\eta_{\widehat{\alpha}\widehat{\alpha}} = \pm 1$) is a frame decomposition of metric (21) on $V^{(\overline{n})}, e_{\widehat{\alpha}}$ is the standard distinguished basis on $\mathcal{R}^{\overline{n}}$, and the projection of torsion , $T_{\mathcal{U}}$, on the base $V^{(\overline{n})}$ is defined as

$$T_{\mathcal{U}} = d\chi_{\mathcal{U}} + \Omega_{\mathcal{U}} \bigwedge \chi_{\mathcal{U}} + \chi_{\mathcal{U}} \bigwedge \Omega_{\mathcal{U}} = e_{\widehat{\alpha}} \otimes \sum_{\overline{\mu}\overline{\nu}} T^{\widehat{\alpha}}_{\overline{\mu}\overline{\nu}} X^{\overline{\mu}} \bigwedge X^{\overline{\nu}}. \tag{52}$$

For a fixed locally adapted basis on $\mathcal{U} \subset V^{(\overline{n})}$ we can identify components $T^{\widehat{a}}_{\overline{\mu}\overline{\nu}}$ of torsion (52) with components of torsion (23) on $V^{(\overline{n})}$, i.e. $T^{\widehat{\alpha}}_{\overline{\mu}\overline{\nu}} = T^{\overline{\alpha}}_{\overline{\mu}\overline{\nu}}$. By straightforward calculation we obtain

$$(\Delta\overline{\mathcal{R}})_{\mathcal{U}} = [(\Delta\mathcal{R}^{(\Gamma)})_{\mathcal{U}},\ (R\tau)_{\mathcal{U}} + (Ri)_{\mathcal{U}}], \tag{53}$$

where

$$(R\tau)_{\mathcal{U}} = \widehat{\delta}_G T_{\mathcal{U}} + *_G^{-1}[\Omega_{\mathcal{U}}, *_G T_{\mathcal{U}}], \quad (Ri)_{\mathcal{U}} = *_G^{-1}\left[\chi_{\mathcal{U}}, *_G \mathcal{R}_{\mathcal{U}}^{(\Gamma)}\right].$$

Form $(Ri)_{\mathcal{U}}$ from (53) is locally constructed by using components of the Ricci tensor (see (25)) as follows from decomposition on the local adapted basis $X^{\overline{\mu}} = \delta u^{\overline{\mu}}$:

$$(Ri)_{\mathcal{U}} = e_{\widehat{\alpha}} \otimes (-1)^{\overline{n}+1} R_{\overline{\lambda}\overline{\nu}} g^{\widehat{\alpha}\overline{\lambda}} \delta u^{\overline{\mu}}.$$

We remark that for isotropic torsionless pseudo-Riemannian spaces the requirement that $\left(\Delta\overline{\mathcal{R}}\right)_{\mathcal{U}}=0$, i.e., imposing the connection (49) to satisfy Yang-Mills equations (40) (equivalently (45) or (46)) we obtain [23, 24] the equivalence of the mentioned gauge gravitational equations with the vacuum Einstein equations $R_{ij}=0$. In the case of ha–spaces with arbitrary given torsion, even considering vacuum gravitational fields, we have to introduce a source for gauge gravitational equations in order to compensate for the contribution of torsion and to obtain equivalence with the Einstein equations.

Considerations presented in this section constitute the proof of the following result:

Theorem 1 *The Einstein equations (27) for ha–gravity are equivalent to the Yang-Mills equations*

$$\left(\Delta\overline{\mathcal{R}}\right)=\overline{\mathcal{J}} \tag{54}$$

for the induced Cartan connection $\overline{\Theta}$ (see (49) and (51)) in the bundle of locally adapted affine frames $\mathcal{A}a\left(V^{(\overline{n})}\right)$ with the source $\overline{\mathcal{J}}_{\mathcal{U}}$ constructed locally by using the same formulas (53) for $\left(\Delta\overline{\mathcal{R}}\right)$, but where $R_{\overline{\alpha}\overline{\beta}}$ is changed by the matter source $E_{\overline{\alpha}\overline{\beta}}-\frac{1}{2}g_{\overline{\alpha}\overline{\beta}}E$ with $E_{\overline{\alpha}\overline{\beta}}=k\Upsilon_{\overline{\alpha}\overline{\beta}}-\lambda g_{\overline{\alpha}\overline{\beta}}$.

We note that this theorem is an extension for higher order anisotropic space-times of the Popov and Dikhin result [24] with respect to a possible gauge like treatment of the Einstein gravity. Similar theorems have been proved for locally anisotropic gauge gravity [33] and in the framework of some variants of locally (and higher order) anisotropic supergravity [30].

6 Nonlinear De Sitter Gauge Ha–Gravity

The equivalent reexpression of the Einstein theory as a gauge like theory implies, for both locally isotropic and anisotropic space–times, the nonsemisimplicity of the gauge group, which leads to a nonvariational theory in the total space of the bundle of locally adapted affine frames. A variational gauge gravitational theory can be formulated by using a minimal extension of the affine structural group $\mathcal{A}f_{\overline{n}}(\mathcal{R})$ to the de Sitter gauge group $S_{\overline{n}}=SO(\overline{n})$ acting on distinguished $\mathcal{R}^{\overline{n}+1}$ space.

6.1 Nonlinear gauge theories of de Sitter group

Let us consider the de Sitter space $\Sigma^{\overline{n}}$ as a hypersurface given by the equations $\eta_{AB}u^Au^B=-l^2$ in the flat $(\overline{n}+1)$–dimensional space enabled with diagonal metric $\eta_{AB},\eta_{AA}=\pm1$ (in this subsection $A,B,C,...=1,2,...,\overline{n}+1$), $(\overline{n}=n+m_1+...+m_z)$, where $\{u^A\}$ are global Cartesian coordinates in $\mathcal{R}^{\overline{n}+1}$; $l>0$ is the curvature of de Sitter space. The de Sitter group $S_{(\eta)}=SO_{(\eta)}(\overline{n}+1)$ is defined as the isometry group of $\Sigma^{\overline{n}}$–space with $\frac{\overline{n}}{2}(\overline{n}+1)$ generators of Lie algebra $so_{(\eta)}(\overline{n}+1)$ satisfying the commutation relations

$$[M_{AB},M_{CD}]=\eta_{AC}M_{BD}-\eta_{BC}M_{AD}-\eta_{AD}M_{BC}+\eta_{BD}M_{AC}. \tag{55}$$

Decomposing indices $A, B, ...$ as $A = (\widehat{\alpha}, \overline{n}+1), B = \left(\widehat{\beta}, \overline{n}+1\right)$, ..., the metric η_{AB} as $\eta_{AB} = \left(\eta_{\widehat{\alpha}\widehat{\beta}}, \eta_{(\overline{n}+1)(\overline{n}+1)}\right)$, and operators M_{AB} as $M_{\widehat{\alpha}\widehat{\beta}} = \mathcal{F}_{\widehat{\alpha}\widehat{\beta}}$ and $P_{\widehat{\alpha}} = l^{-1} M_{\overline{n}+1, \widehat{\alpha}}$, we can write (55) as

$$\left[\mathcal{F}_{\widehat{\alpha}\widehat{\beta}}, \mathcal{F}_{\widehat{\gamma}\widehat{\delta}}\right] = \eta_{\widehat{\alpha}\widehat{\gamma}} \mathcal{F}_{\widehat{\beta}\widehat{\delta}} - \eta_{\widehat{\beta}\widehat{\gamma}} \mathcal{F}_{\widehat{\alpha}\widehat{\delta}} + \eta_{\widehat{\beta}\widehat{\delta}} \mathcal{F}_{\widehat{\alpha}\widehat{\gamma}} - \eta_{\widehat{\alpha}\widehat{\delta}} \mathcal{F}_{\widehat{\beta}\widehat{\gamma}},$$

$$\left[P_{\widehat{\alpha}}, P_{\widehat{\beta}}\right] = -l^{-2} \mathcal{F}_{\widehat{\alpha}\widehat{\beta}}, \quad \left[P_{\widehat{\alpha}}, \mathcal{F}_{\widehat{\beta}\widehat{\gamma}}\right] = \eta_{\widehat{\alpha}\widehat{\beta}} P_{\widehat{\gamma}} - \eta_{\widehat{\alpha}\widehat{\gamma}} P_{\widehat{\beta}},$$

where we have indicated the possibility to decompose $so_{(\eta)}(\overline{n}+1)$ into a direct sum, $so_{(\eta)}(\overline{n}+1) = so_{(\eta)}(\overline{n}) \oplus v_{\overline{n}}$, where $v_{\overline{n}}$ is the vector space stretched on vectors $P_{\widehat{\alpha}}$. We remark that $\Sigma^{\overline{n}} = S_{(\eta)}/L_{(\eta)}$, where $L_{(\eta)} = SO_{(\eta)}(\overline{n})$. For $\eta_{AB} = diag\,(1, -1, -1, -1)$ and $S_{10} = SO(1,4), L_6 = SO(1,3)$ is the group of Lorentz rotations.

Let $W\left(\mathcal{E}, \mathcal{R}^{\overline{n}+1}, S_{(\eta)}, \mathcal{P}\right)$ be the vector bundle associated with the principal bundle $\mathcal{P}\left(S_{(\eta)}, \mathcal{E}\right)$ on ha-spacetime $v_{\overline{n}}$, where $S_{(\eta)}$ is taken to be the structural group and by $\mathcal{E}$ it is denoted the total space. The action of the structural group $S_{(\eta)}$ on $\mathcal{E}$ can be realized by using $\overline{n} \times \overline{n}$ matrices with a parametrization distinguishing subgroup $L_{(\eta)}$:

$$B = bB_L, \tag{56}$$

where

$$B_L = \left(\begin{array}{cc} L & 0 \\ 0 & 1 \end{array}\right),$$

$L \in L_{(\eta)}$ is the de Sitter bust matrix transforming the vector $(0, 0, ..., \rho) \in \mathcal{R}^{\overline{n}+1}$ into the point $\left(v^1, v^2, ..., v^{\overline{n}+1}\right) \in \Sigma^{\overline{n}}_{\rho} \subset \mathcal{R}^{\overline{n}+1}$ for which $v_A v^A = -\rho^2, v^A = t^A \rho$. Matrix b can be expressed

$$b = \left(\begin{array}{cc} \delta^{\widehat{\alpha}}{}_{\widehat{\beta}} + \frac{t^{\widehat{\alpha}} t_{\widehat{\beta}}}{(1+t^{\overline{n}+1})} & t^{\widehat{\alpha}} \\ t_{\widehat{\beta}} & t^{\overline{n}+1} \end{array}\right).$$

The de Sitter gauge field is associated with a linear connection in W, i.e., with a $so_{(\eta)}(\overline{n}+1)$-valued connection 1–form on $V^{(\overline{n})}$:

$$\breve{\Theta} = \left(\begin{array}{cc} \omega^{\widehat{\alpha}}{}_{\widehat{\beta}} & \breve{\theta}^{\widehat{\alpha}} \\ \breve{\theta}_{\widehat{\beta}} & 0 \end{array}\right), \tag{57}$$

where $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}} \in so(\overline{n})_{(\eta)}, \breve{\theta}^{\widehat{\alpha}} \in \mathcal{R}^{\overline{n}}, \breve{\theta}_{\widehat{\beta}} \in \eta_{\widehat{\beta}\widehat{\alpha}} \breve{\theta}^{\widehat{\alpha}}$.

Because $S_{(\eta)}$-transforms mix $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $\breve{\theta}^{\widehat{\alpha}}$ fields in (57) (the introduced parametrization is invariant on action on $SO_{(\eta)}(\overline{n})$ group we cannot identify $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $\breve{\theta}^{\widehat{\alpha}}$, respectively, with the connection $\Gamma^{\overline{\alpha}}{}_{\overline{\beta}\overline{\gamma}}$ and the fundamental form $\chi^{\overline{\alpha}}$ in $V^{(\overline{n})}$ (as we have for (49) and (51)). To avoid this difficulty we consider [34, 22] a nonlinear gauge realization of the de Sitter group $S_{(\eta)}$ by introducing the nonlinear gauge field

$$\Theta = b^{-1} \breve{\Theta} b + b^{-1} db = \left(\begin{array}{cc} \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & \theta^{\widehat{\alpha}} \\ \theta_{\widehat{\beta}} & 0 \end{array}\right), \tag{58}$$

where

$$\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} = \omega^{\widehat{\alpha}}{}_{\widehat{\beta}} - \left(t^{\widehat{\alpha}} Dt_{\widehat{\beta}} - t_{\widehat{\beta}} Dt^{\widehat{\alpha}}\right) / \left(1 + t^{\overline{n}+1}\right),$$

$$\theta^{\widehat{\alpha}} = t^{\overline{n}+1} \breve{\theta}^{\widehat{\alpha}} + Dt^{\widehat{\alpha}} - t^{\widehat{\alpha}} \left(dt^{\overline{n}+1} + \breve{\theta}_{\widehat{\gamma}} t^{\widehat{\gamma}}\right) / \left(1 + t^{\overline{n}+1}\right),$$

$$Dt^{\widehat{\alpha}} = dt^{\widehat{\alpha}} + \omega^{\widehat{\alpha}}{}_{\widehat{\beta}} t^{\widehat{\beta}}.$$

The action of the group $S(\eta)$ is nonlinear, yielding transforms

$$\Gamma' = L'\Gamma \left(L'\right)^{-1} + L'd\left(L'\right)^{-1}, \theta' = L\theta,$$

where the nonlinear matrix-valued function $L' = L'\left(t^{\overline{\alpha}}, b, B_T\right)$ is defined from $B_b = b' B_{L'}$ (see the parametrization (56)).

Now, we can identify components of (58) with components of $\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\gamma}}$ and $\chi^{\widehat{\alpha}}_{\overline{\alpha}}$ on $V^{(\overline{n})}$ and induce in a consistent manner on the base of bundle $W(\mathcal{E}, \mathcal{R}^{\overline{n}+1}, S_{(\eta)}, \mathcal{P})$ the ha–geometry.

6.2 Dynamics of the nonlinear de Sitter ha–gravity

Instead of the gravitational potential (49), we introduce the gravitational connection (similar to (58))

$$\Gamma = \left(\begin{array}{cc} \Gamma^{\widehat{\alpha}}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ l_0^{-1}\chi_{\widehat{\beta}} & 0 \end{array} \right) \tag{59}$$

where

$$\Gamma^{\widehat{\alpha}}_{\widehat{\beta}} = \Gamma^{\widehat{\alpha}}_{\widehat{\beta}\overline{\mu}} \delta u^{\overline{\mu}},$$

$$\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}} = \chi^{\widehat{\alpha}}{}_{\overline{\alpha}} \chi^{\widehat{\beta}}{}_{\overline{\beta}} \Gamma^{\overline{\alpha}}{}_{\overline{\beta}\overline{\gamma}} + \chi^{\widehat{\alpha}}{}_{\overline{\alpha}} \delta_{\overline{\mu}} \chi^{\overline{\alpha}}{}_{\widehat{\beta}},$$

$\chi^{\widehat{\alpha}} = \chi^{\widehat{\alpha}}{}_{\overline{\mu}} \delta u^{\overline{\mu}}$, and $g_{\overline{\alpha}\overline{\beta}} = \chi^{\widehat{\alpha}}{}_{\overline{\alpha}} \chi^{\widehat{\beta}}{}_{\overline{\beta}} \eta_{\widehat{\alpha}\widehat{\beta}}$, and $\eta_{\widehat{\alpha}\widehat{\beta}}$ is parametrized as

$$\eta_{\widehat{\alpha}\widehat{\beta}} = \left(\begin{array}{cccc} \eta_{ij} & 0 & ... & 0 \\ 0 & \eta_{a_1 b_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \eta_{a_z b_z} \end{array} \right),$$

$\eta_{ij} = (1, -1, ..., -1), ...\eta_{ij} = (\pm 1, \pm 1, ..., \pm 1), ..., l_0$ is a dimensional constant.

The curvature of (59), $\mathcal{R}^{(\Gamma)} = d\Gamma + \Gamma \bigwedge \Gamma$, can be written as

$$\mathcal{R}^{(\Gamma)} = \left(\begin{array}{cc} \mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}} + l_0^{-1}\pi^{\widehat{\alpha}}_{\widehat{\beta}} & l_0^{-1}T^{\widehat{\alpha}} \\ l_0^{-1}T^{\widehat{\beta}} & 0 \end{array} \right), \tag{60}$$

where

$$\pi^{\widehat{\alpha}}_{\widehat{\beta}} = \chi^{\widehat{\alpha}} \bigwedge \chi_{\widehat{\beta}}, \mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}} = \frac{1}{2} \mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}\overline{\nu}} \delta u^{\overline{\mu}} \bigwedge \delta u^{\overline{\nu}},$$

and

$$\mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}\overline{\nu}} = \chi_{\widehat{\beta}}{}^{\overline{\beta}} \chi_{\overline{\alpha}}{}^{\widehat{\alpha}} R^{\overline{\alpha}}_{\overline{\beta}.\overline{\mu}\overline{\nu}}$$

(see (24) for components of d-curvatures). The de Sitter gauge group is semisimple and we are able to construct a variational gauge gravitational locally anisotropic theory (bundle metric (37) is nondegenerate). The Lagrangian of the theory is postulated as

$$L = L_{(G)} + L_{(m)}$$

where the gauge gravitational Lagrangian is defined as

$$L_{(G)} = \frac{1}{4\pi} Tr \left(\mathcal{R}^{(\Gamma)} \bigwedge *_G \mathcal{R}^{(\Gamma)} \right) = \mathcal{L}_{(G)} \left| g \right|^{1/2} \delta^{\overline{n}} u,$$

$$\mathcal{L}_{(G)} = \frac{1}{2l^2} T^{\widehat{\alpha}}{}_{\overline{\mu\nu}} T_{\widehat{\alpha}}{}^{\overline{\mu\nu}} + \frac{1}{8\lambda} \mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu\nu}} \mathcal{R}^{\widehat{\beta}}{}_{\widehat{\alpha}}{}^{\overline{\mu\nu}} - \frac{1}{l^2} \left(\overleftarrow{R} \left(\Gamma \right) - 2\lambda_1 \right), \tag{61}$$

$T^{\widehat{\alpha}}{}_{\overline{\mu\nu}} = \chi^{\widehat{\alpha}}{}_{\overline{\alpha}} T^{\overline{\alpha}}{}_{\overline{\mu\nu}}$ (the gravitational constant l^2 in (61) satisfies the relations $l^2 = 2l_0^2 \lambda, \lambda_1 = -3/l_0],$ Tr denotes the trace on $\widehat{\alpha}, \widehat{\beta}$ indices, and the matter field Lagrangian is defined as

$$L_{(m)} = \frac{1}{2} Tr \left(\Gamma \bigwedge *_G \mathcal{I} \right) = \mathcal{L}_{(m)} \left| g \right|^{1/2} \delta^{\overline{n}} u,$$

$$\mathcal{L}_{(m)} = \frac{1}{2} \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}} S^{\widehat{\beta}}{}_{\widehat{\alpha}}{}^{\overline{\mu}} - t^{\overline{\mu}}{}_{\widehat{\alpha}} l^{\widehat{\alpha}}{}_{\overline{\mu}}. \tag{62}$$

The matter field source $\mathcal{I}$ is obtained as a variational derivation of $\mathcal{L}_{(m)}$ on Γ and is parametrized as

$$\mathcal{I} = \begin{pmatrix} S^{\widehat{\alpha}}{}_{\widehat{\beta}} & -l_0 t^{\widehat{\alpha}} \\ -l_0 t_{\widehat{\beta}} & 0 \end{pmatrix} \tag{63}$$

with $t^{\widehat{\alpha}} = t^{\widehat{\alpha}}{}_{\overline{\mu}} \delta u^{\overline{\mu}}$ and $S^{\widehat{\alpha}}{}_{\widehat{\beta}} = S^{\widehat{\alpha}}{}_{\widehat{\beta}\overline{\mu}} \delta u^{\overline{\mu}}$ being respectively the canonical tensors of energy-momentum and spin density. Because of the contraction of the "interior" indices $\widehat{\alpha}, \widehat{\beta}$ in (61) and (62) we used the Hodge operator $*_G$ instead of $*_H$ (hereafter we consider $*_G = *$).

Varying the action

$$S = \int \left| g \right|^{1/2} \delta^{\overline{n}} u \left(\mathcal{L}_{(G)} + \mathcal{L}_{(m)} \right)$$

on the Γ-variables (59), we obtain the gauge–gravitational field equations:

$$d \left(* \mathcal{R}^{(\Gamma)} \right) + \Gamma \bigwedge \left(* \mathcal{R}^{(\Gamma)} \right) - \left(* \mathcal{R}^{(\Gamma)} \right) \bigwedge \Gamma = -\lambda \left(* \mathcal{I} \right). \tag{64}$$

Specifying the variations on $\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $l^{\widehat{\alpha}}$-variables, we rewrite (64) as

$$\widehat{\mathcal{D}} \left(* \mathcal{R}^{(\Gamma)} \right) + \frac{2\lambda}{l^2} \left(\widehat{\mathcal{D}} \left(* \pi \right) + \chi \bigwedge \left(* T^T \right) - \left(* T \right) \bigwedge \chi^T \right) = -\lambda \left(* S \right), \tag{65}$$

$$\widehat{\mathcal{D}} \left(* T \right) - \left(* \mathcal{R}^{(\Gamma)} \right) \bigwedge \chi - \frac{2\lambda}{l^2} \left(* \pi \right) \bigwedge \chi = \frac{l^2}{2} \left(* t + \frac{1}{\lambda} * \tau \right), \tag{66}$$

where

$$T^t = \{ T_{\widehat{\alpha}} = \eta_{\widehat{\alpha}\widehat{\beta}} T^{\widehat{\beta}}, \; T^{\widehat{\beta}} = \frac{1}{2} T^{\widehat{\beta}}{}_{\overline{\mu\nu}} \delta u^{\overline{\mu}} \bigwedge \delta u^{\overline{\nu}} \},$$

$$\chi^T = \{\chi_{\widehat{\alpha}} = \eta_{\widehat{\alpha}\widehat{\beta}}\chi^{\widehat{\beta}},\ \chi^{\widehat{\beta}} = \chi^{\widehat{\beta}}\ _{\overline{\mu}}\delta u^{\overline{\mu}}\}, \qquad \widehat{\mathcal{D}} = d + \widehat{\Gamma}$$

($\widehat{\Gamma}$ acts as $\Gamma^{\widehat{\alpha}}\ _{\widehat{\beta}\overline{\mu}}$ on indices $\widehat{\gamma}, \widehat{\delta}, ...$ and as $\Gamma^{\overline{\alpha}}\ _{\overline{\beta}\overline{\mu}}$ on indices $\overline{\gamma},\ \overline{\delta}, ...$). In (66), τ defines the energy–momentum tensor of the $S_{(\eta)}$–gauge gravitational field $\widehat{\Gamma}$:

$$\tau_{\overline{\mu}\overline{\nu}}\left(\widehat{\Gamma}\right) = \frac{1}{2}Tr\left(\mathcal{R}_{\overline{\mu}\overline{\alpha}}\mathcal{R}^{\overline{\alpha}}\ _{\overline{\nu}} - \frac{1}{4}\mathcal{R}_{\overline{\alpha}\overline{\beta}}\mathcal{R}^{\overline{\alpha}\overline{\beta}}g_{\overline{\mu}\overline{\nu}}\right). \tag{67}$$

Equations (64) (or, equivalently, (65) and (66)) make up the complete system of variational field equations for nonlinear de Sitter gauge gravity with higher order anisotropy. They can be interpreted as a variant of gauge like equations for ha–gravity [33] when the (pseudo) Riemannian base frames and torsions are considered to be induced by an anholonomic frame structure with associated N–connection

A. Tseytlin [34] presented a quantum analysis of the isotropic version of equations (65) and (66). Of course, the problem of quantizing gravitational interactions is unsolved for both variants of locally anisotropic and isotropic gauge de Sitter gravitational theories, but we think that the generalized Lagrange version of $S_{(\eta)}$-gravity is more adequate for studying quantum radiational and statistical gravitational processes. This is a matter for further investigations.

Finally, we remark that we can obtain a nonvariational Poincare gauge gravitational theory on ha–spaces if we consider the contraction of the gauge potential (59) to a potential with values in the Poincare Lie algebra

$$\Gamma = \begin{pmatrix} \Gamma^{\widehat{\alpha}}\ _{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ l_0^{-1}\chi_{\widehat{\beta}} & 0 \end{pmatrix} \rightarrow \Gamma = \begin{pmatrix} \Gamma^{\widehat{\alpha}}\ _{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ 0 & 0 \end{pmatrix}.$$

Isotropic Poincare gauge gravitational theories are studied in a number of papers (see, for example, [35, 34, 22]). In a manner similar to considerations presented in this work, we can generalize Poincare gauge models for spaces with local anisotropy.

7 An Ansatz for 4D d–Metrics

We consider a 4D spacetime $V^{(3+1)}$ provided with a d–metric (8) when $g_i = g_i(x^k)$ and $h_a = h_a(x^k, z)$ for $y^a = (z, y^4)$. The N–connection coefficients are some functions on three coordinates (x^i, z),

$$\begin{aligned} N_1^3 &= q_1(x^i, z),\ N_2^3 = q_2(x^i, z), \\ N_1^4 &= n_1(x^i, z),\ N_2^4 = n_2(x^i, z). \end{aligned} \tag{68}$$

For simplicity, we shall use brief denotations of partial derivatives, like $\dot{a}= \partial a/\partial x^1, a' = \partial a/\partial x^2$, $a^* = \partial a/\partial z$ $\dot{a}'= \partial^2 a/\partial x^1\partial x^2$, $a^{**} = \partial^2 a/\partial z\partial z$.

The non–trivial components of the Ricci d–tensor (11), for the mentioned type of d–metrics depending on three variables, are

$$R_1^1 = R_2^2 = \frac{1}{2g_1g_2}[-(g_1^{''} + \ddot{g}_2) + \frac{1}{2g_2}\left(\dot{g}_2^2 + g_1'g_2'\right) + \frac{1}{2g_1}\left(g_1'^{\,2} + \dot{g}_1\dot{g}_2\right)]; \tag{69}$$

$$
\begin{aligned}
S_3^3 &= S_4^4 = \frac{1}{h_3 h_4}[-h_4^{**} + \frac{1}{2h_4}(h_4^*)^2 + \frac{1}{2h_3} h_3^* h_4^*]; && (70)\\
P_{31} &= \frac{q_1}{2}[\left(\frac{h_3^*}{h_3}\right)^2 - \frac{h_3^{**}}{h_3} + \frac{h_4^*}{2h_4{}^2} - \frac{h_3^* h_4^*}{2h_3 h_4}] + \frac{1}{2h_4}[\frac{\dot{h}_4}{2h_4} h_4^* - \dot{h}_4^* + \frac{\dot{h}_3}{2h_3} h_4^*], && (71)\\
P_{32} &= \frac{q_2}{2}[\left(\frac{h_3^*}{h_3}\right)^2 - \frac{h_3^{**}}{h_3} + \frac{h_4^*}{2h_4{}^2} - \frac{h_3^* h_4^*}{2h_3 h_4}] + \frac{1}{2h_4}[\frac{h_4'}{2h_4} h_4^* - h_4'{}^* + \frac{h_3'}{2h_3} h_4^*]; &&\\
P_{41} &= -\frac{h_4}{2h_3} n_1^{**} + \frac{1}{4h_3}(\frac{h_4}{h_3} h_3^* - 3h_4^*) n_1^*, && (72)\\
P_{42} &= -\frac{h_4}{2h_3} n_2^{**} + \frac{1}{4h_3}(\frac{h_4}{h_3} h_3^* - 3h_4^*) n_2^*. &&
\end{aligned}
$$

The curvature scalar $\overleftarrow{R}$ (12) is defined by the sum of two non-trivial components $\widehat{R} = 2R_1^1$ and $S = 2S_3^3$.

The system of Einstein equations (13) transforms into

$$
\begin{aligned}
R_1^1 &= -\kappa \Upsilon_3^3 = -\kappa \Upsilon_4^4, && (73)\\
S_3^3 &= -\kappa \Upsilon_1^1 = -\kappa \Upsilon_2^2, && (74)\\
P_{3i} &= \kappa \Upsilon_{3i}, && (75)\\
P_{4i} &= \kappa \Upsilon_{4i}, && (76)
\end{aligned}
$$

where the values of R_1^1, S_3^3, P_{ai}, are taken respectively from (69), (70), (71), (72).

By using the equations (75) and (76) we can define the N–coefficients (68), $q_i(x^k, z)$ and $n_i(x^k, z)$, if the functions $g_i(x^k)$ and $h_i(x^k, z)$ are known as respective solutions of the equations (73) and (74).

Let consider an ansatz for a 4D d–metric of type

$$
\delta s^2 = g_1(x^k)(dx^1)^2 + (dx^2)^2 + h_3(x^i, t)(\delta t)^2 + h_4(x^i, t)(\delta y^4)^2, \qquad (77)
$$

where the z–parameter is considered to be the time like coordinate and the energy momentum d–tensor is taken

$$
\Upsilon_\alpha^\beta = [p_1, p_2, -\varepsilon, p_4 = p].
$$

The aim of this section is to analyze the system of partial differential equations forllowing form the Einsteni field equations for these d–metric and energy–momentum d–tensor.

7.1 The h–equations

The Einstein equations (73), with the Ricci h–tensor (69), for the d–metric (77) transform into

$$
\frac{\partial^2 g_1}{\partial (x^1)^2} - \frac{1}{2g_1}\left(\frac{\partial g_1}{\partial x^1}\right)^2 + 2\kappa\varepsilon g_1 = 0. \qquad (78)
$$

By introducing the coordinates $\chi^i = x^i/\sqrt{\kappa\varepsilon}$ and the variable

$$
q = g_1'/g_1, \qquad (79)
$$

where by 'prime' in this Section is considered the partial derivative ∂/χ^2, the equation (78) transforms into

$$q' + \frac{q^2}{2} + 2\epsilon = 0, \qquad (80)$$

where the vacuum case should be parametrized for $\epsilon = 0$ with $\chi^i = x^i$ and $\epsilon = -1$ for a matter state with $\varepsilon = -p$.

The integral curve of (80), intersecting a point $\left(\chi^2_{(0)}, q_{(0)}\right)$, considered as a differential equation on χ^2 is defined by the functions [11]

$$q = \frac{q_{(0)}}{1 + \frac{q_{(0)}}{2}\left(\chi^2 - \chi^2_{(0)}\right)}, \qquad \epsilon = 0; \qquad (81)$$

$$q = \frac{q_{(0)} - 2\tan\left(\chi^2 - \chi^2_{(0)}\right)}{1 + \frac{q_{(0)}}{2}\tan\left(\chi^2 - \chi^2_{(0)}\right)}, \qquad \epsilon < 0. \qquad (82)$$

Because the function q depends also parametrically on variable χ^1 we can consider functions $\chi^2_{(0)} = \chi^2_{(0)}\left(\chi^1\right)$ and $q_{(0)} = q_{(0)}\left(\chi^1\right)$.

We elucidate the nonvacuum case with $\epsilon < 0$. The general formula for the non–trivial component of h–metric is to be obtained after integration on χ^1 of (79) by using the solution (82)

$$g_1\left(\chi^1, \chi^2\right) = g_{1(0)}\left(\chi^1\right)\left\{\sin[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)] + \arctan\frac{2}{q_{(0)}\left(\chi^1\right)}\right\}^2,$$

for $q_{(0)}\left(\chi^1\right) \neq 0$, and

$$g_1\left(\chi^1, \chi^2\right) = g_{1(0)}\left(\chi^1\right)\ \cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)] \qquad (83)$$

for $q_{(0)}(\chi^1) = 0$, where $g_{1(0)}(\chi^1), \chi^2_{(0)}(\chi^1)$ and $q_{(0)}(\chi^1)$ are some functions of necessary smoothness class on variable χ^1.

For simplicity, in our further considerations we shall apply the solution (83).

7.2 The v–equations

For the ansatz (77) the Einstein equations (74) with the Ricci h–tensor (70) transforms into

$$\frac{\partial^2 h_4}{\partial t^2} - \frac{1}{2h_4}\left(\frac{\partial h_4}{\partial t}\right)^2 - \frac{1}{2h_3}\left(\frac{\partial h_3}{\partial t}\right)\left(\frac{\partial h_4}{\partial t}\right) - \frac{\kappa}{2}\Upsilon_1 h_3 h_4 = 0 \qquad (84)$$

(here we write down the partial derivatives on t in explicit form) which relates some first and second order partial on z derivatives of diagonal components $h_a(x^i, t)$ of a v–metric with a source

$$\Upsilon_1(x^i, z) = \kappa\Upsilon^1_1 = \kappa\Upsilon^2_2 = p_1 = p_2$$

in the h–subspace. We can consider as unknown the function $h_3(x^i, t)$ (or, inversely, $h_4(x^i, t)$) for some compatible values of $h_4(x^i, t)$ (or $h_3(x^i, t)$) and source

$\Upsilon_1(x^i,t)$. By introducing a new variable $\beta = h_4^*/h_4$ the equation (84) transforms into
$$\beta^* + \frac{1}{2}\beta^2 - \frac{\beta h_3^*}{2h_3} - 2\kappa\Upsilon_1 h_3 = 0 \tag{85}$$
which relates two functions $\beta\left(x^i,t\right)$ and $h_3\left(x^i,t\right)$. There are two possibilities: 1) to define β (i. e. h_4) when $\kappa\Upsilon_1$ and h_3 are prescribed and, inversely 2) to find h_3 for given $\kappa\Upsilon_1$ and h_4 (i. e. β); in both cases one considers only "*" derivatives on t–variable with coordinates x^i treated as parameters.

1. In the first case the explicit solutions of (85) have to be constructed by using the integral varieties of the general Riccati equation [11] which by a corresponding redefinition of variables, $t \to t(\varsigma)$ and $\beta(t) \to \eta(\varsigma)$ (for simplicity, we omit dependencies on x^i) could be written in the canonical form
$$\frac{\partial \eta}{\partial \varsigma} + \eta^2 + \Psi(\varsigma) = 0$$
where Ψ vanishes for vacuum gravitational fields. In vacuum cases the Riccati equation reduces to a Bernoulli equation which (we can use the former variables) for $s(t) = \beta^{-1}$ transforms into a linear differential (on t) equation,
$$s^* + \frac{h_3^*}{2h_3}s - \frac{1}{2} = 0. \tag{86}$$

2. In the second (inverse) case when h_3 is to be found for some prescribed $\kappa\Upsilon_1$ and β the equation (85) is to be treated as a Bernoulli type equation,
$$h_3^* = -\frac{4\kappa\Upsilon_1}{\beta}(h_3)^2 + \left(\frac{2\beta^*}{\beta} + \beta\right)h_3 \tag{87}$$
which can be solved by standard methods. In the vacuum case the squared on h_3 term vanishes and we obtain a linear differential (on t) equation.

Finally, in this Section we conclude that the system of equations (74) is satisfied by arbitrary functions
$$h_3 = a_3(\chi^i) \text{ and } h_4 = a_4(\chi^i).$$
If v–metrics depending on three coordinates are introduced, $h_a = h_a(\chi^i,t)$, the v–components of the Einstein equations transforms into (84) which reduces to (85) for prescribed values of $h_3(\chi^i,t),$ and, inversely, to (87) if $h_4(\chi^i,t)$ is prescribed.

7.3 H–v equations

For the ansatz (77) with $h_4 = h_4(x^i)$ and a diagonal energy–momentum d–tensor the h–v–components of Einstein equations (75) and (76) are written respectively as
$$P_{5i} = \frac{q_i}{2h_3}[(\frac{\partial h_3}{\partial t})^2 - \frac{\partial^2 h_3}{\partial t^2}] = 0, \tag{88}$$
and
$$P_{6i} = \frac{h_4}{4(h_3)^2}\frac{\partial n_i}{\partial t}\frac{\partial h_3}{\partial t} - \frac{h_4}{2h_3}\frac{\partial^2 n_i}{\partial t^2} = 0. \tag{89}$$

The equations (88) are satisfied by arbitrary coefficients $q_i(x^k, t)$ if the d–metric coefficient h_3 is a solution of

$$(\frac{\partial h_3}{\partial t})^2 - \frac{\partial^2 h_3}{\partial t^2} = 0 \tag{90}$$

and the q–coefficients must vanish if this condition is not satisfied. In the last case we obtain a $3+1$ anisotropy.

The general solution of equations (89) are written in the form

$$n_i = l_i^{(0)}(x^k) \int \sqrt{|h_3(x^k, t)|}dt + n_i^{(0)}(x^k)$$

where $l_i^{(0)}(x^k)$ and $n_i^{(0)}(x^k)$ are arbitrary functions on x^k which have to be defined by some boundary conditions.

8 Cosmological La–Solutions

The aim of this section is to construct two classes of solutions of Einstein equations describing Friedman–Robertson–Walker (FRW) like universes with corresponding symmetries or rotational ellipsoid (ellongated and flattend) and torus.

8.1 Rotation ellipsoid FRW universes

We proof that there are cosmological solutions constructed as locally anisotropic deformations of the FRW spherical symmetric solution to the rotation ellipsoid configuration. There are two types of rotation ellipsoids, elongated and flattened ones. We examine both cases of such horizon configurations.

8.1.1 Rotation elongated ellipsoid configuration

An elongated rotation ellipsoid hypersurface is given by the formula [12]

$$\frac{x^2 + y^2}{\sigma^2 - 1} + \frac{z^2}{\sigma^2} = \rho^2, \tag{91}$$

where $\sigma \geq 1$, x, y, z are Cartezian coordinates and ρ is similar to the radial coordinate in the spherical symmetric case.

The 3D special coordinate system is defined

$$\begin{aligned} x &= \rho \sinh u \sin v \cos \varphi, \ y = \rho \sinh u \sin v \sin \varphi, \\ z &= \rho \ \cosh u \cos v, \end{aligned}$$

where $\sigma = \cosh u, (0 \leq u < \infty, \ 0 \leq v \leq \pi, \ 0 \leq \varphi < 2\pi)$. The hypersurface metric (91) is

$$\begin{aligned} g_{uu} &= g_{vv} = \rho^2 \left(\sinh^2 u + \sin^2 v\right), \\ g_{\varphi\varphi} &= \rho^2 \sinh^2 u \sin^2 v. \end{aligned} \tag{92}$$

Let us introduce a d–metric of class (77)

$$\delta s^2 = g_1(u, v)du^2 + dv^2 + h_3(u, v, \tau)(\delta\tau)^2 + h_4(u, v)(\delta\varphi)^2, \tag{93}$$

where $x^1 = u, x^2 = v$, $y^4 = \varphi$, $y^3 = \tau$ is the time like cosmological coordinate and $\delta\tau$ and $\delta\varphi$ are N–elongated differentials.

As a particular solution of (93) for the h–metric we choose (see (83)) the coefficient

$$g_1(u, v) = \cos^2 v \tag{94}$$

and set for the v–metric components

$$h_3(u, v, \tau) = -\frac{1}{\rho^2(\tau)(\sinh^2 u + \sin^2 v)} \tag{95}$$

and

$$h_4(u, v, \tau) = \frac{\sinh^2 u \sin^2 v}{(\sinh^2 u + \sin^2 v)}. \tag{96}$$

The set of coefficients (94),(95), and (96), for the d–metric (93, and of $q_i = 0$ and n_i being solutions of (90), for the N–connection, defines a solution of the Einstein equations (13).

The physical treatment of the obtained solutions follows from the locally isotropic limit of a conformal transform of this d–metric: Multiplying (93) on

$$\rho^2(\tau)(\sinh^2 u + \sin^2 v),$$

and considering $cos^2 v \simeq 1$ and $n_i \simeq= 0$ for locally isotropic spacetimes we get the interval

$$\begin{aligned} ds^2 &= -d\tau^2 + \rho^2(\tau)[(\sinh^2 u + \sin^2 v)(du^2 + dv^2) + \sinh^2 u \sin^2 v d\varphi^2] \\ &\quad \text{for ellipsoidal coordinates on hypersurface (92);} \\ &= -d\tau^2 + \rho^2(\tau)[dx^2 + dy^2 + dz^2] \text{ for Cartezian coordinates,} \end{aligned}$$

which defines just the Robertson–Walker metric.

So, the d–metric (93), the coefficients of N–connection being solutions of (75) and (76), describes a 4D cosmological solution of the Einstein equations when instead of a spherical symmetry one has a locally anisotropic deformation to the symmetry of rotation elongated ellipsoid. The explicit dependence on time τ of the cosmological factor ρ must be constructed by using additionally the matter state equations for a cosmological model with local anisotropy.

8.1.2 Flattened rotation ellipsoid coordinates

In a similar fashion we can construct a locally anisotropic deformation of the FRW metric with the symmetry of flattened rotation ellipsoid. The parametric equation for a such hypersurface is [12]

$$\frac{x^2 + y^2}{1 + \sigma^2} + \frac{z^2}{\sigma^2} = \rho^2,$$

where $\sigma \geq 0$ and $\sigma = \sinh u$.

The proper for ellipsoid 3D space coordinate system is defined

$$\begin{aligned} x &= \rho \cosh u \sin v \cos \varphi, y = \rho \cosh u \sin v \sin \varphi \\ z &= \rho \sinh u \cos v, \end{aligned}$$

where $0 \leq u < \infty,\ 0 \leq v \leq \pi,\ 0 \leq \varphi < 2\pi.$

The hypersurface metric is

$$\begin{aligned} g_{uu} &= g_{vv} = \rho^2 \left(\sinh^2 u + \cos^2 v\right), \\ g_{\varphi\varphi} &= \rho^2 \sinh^2 u \cos^2 v. \end{aligned}$$

In the rest the cosmological la–solution is described by the same formulas as in the previous subsection but with respect to new canonical coordinates for flattened rotation ellipsoid.

8.2 Toroidal FRW universes

Let us construct a cosmological solution of the Einstein equations with toroidal symmetry. The hypersurface formula of a torus is [12]

$$\left(\sqrt{x^2+y^2} - \rho\ c\tanh\sigma\right)^2 + z^2 = \frac{\rho^2}{\sinh^2\sigma}.$$

The 3D space coordinate system is defined

$$\begin{aligned} x &= \frac{\rho \sinh\alpha\cos\varphi}{\cosh\alpha - \cos\sigma}, \qquad y = \frac{\rho\sin\sigma\sin\varphi}{\cosh\alpha - \cos\sigma}, \\ z &= \frac{\rho\sinh\sigma}{\cosh\tau - \cos\sigma}, \\ & \left(-\pi < \sigma < \pi, 0 \leq \alpha < \infty, 0 \leq \varphi < 2\pi\right). \end{aligned}$$

The hypersurface metric is

$$g_{\sigma\sigma} = g_{\alpha\alpha} = \frac{\rho^2}{(\cosh\alpha - \cos\sigma)^2}, g_{\varphi\varphi} = \frac{\rho^2\sin^2\sigma}{(\cosh\alpha-\cos\sigma)^2}. \tag{97}$$

The d–metric of class (77) is chosen

$$\delta s^2 = g_1(\alpha)d\sigma^2 + d\alpha^2 + h_3\left(\sigma,\alpha,\tau\right)\left(\delta\tau\right)^2 + h_4\left(\sigma\right)\left(\delta\varphi\right)^2, \tag{98}$$

where $x^1 = \sigma, x^2 = \alpha,\ y^4 = \varphi,\ y^3 = \tau$ is the time like cosmological coordinate and $\delta\tau$ and $\delta\varphi$ are N–elongated differentials.

As a particular solution of (97) for the h–metric we choose (see (83)) the coefficient

$$g_1(\alpha) = \cos^2\alpha \tag{99}$$

and set for the v–metric components

$$\begin{aligned} h_3(\sigma,\alpha,\tau) &= -\frac{(\cosh\alpha-\cos\sigma)^2}{\rho^2(\tau)} \\ h_4(\sigma) &= \sin^2\sigma. \end{aligned} \tag{100}$$

Multiplying (98) on

$$\frac{\rho^2(\tau)}{(\cosh\alpha-\cos\sigma)^2},$$

and considering $cos\alpha \simeq 1$ and $n_i \simeq= 0$ in the locally isotropic limit we get the interval

$$ds^2 = -d\tau^2 + \frac{\rho^2(\tau)}{(\cosh\alpha - \cos\sigma)^2}[(d\sigma^2 + d\alpha^2 + \sin^2\sigma d\varphi^2]$$

where the space part is just the torus hypersurface metric (97).

So, the set of coefficients (99) and (100), for the d–metric (98, and of $q_i = 0$ and n_i being solutions of (90), for the N–connection, defines a cosmologica solution of the Einstein equations (13) with the torus symmetry, when the explicit form of the function $\rho(\tau)$ is to be defined by considering some additional equations for the matter state (for instance, with a scalar field defining the torus inflation).

9 Outlook and Concluding Remarks

In this paper we have developed the method of anholonomic frames on (pseudo) Riemannian spacetimes by considering associated nonlinear connection (N–connection) strucutres. We provided a rigorous geometric background for description of gravitational systems with mixed holonomic and anholonomic (anisotropic) degrees of freedom by considering first and higher order anisotropies induced by anholonomic constraints and corresponding frame bases.

The first key result of this paper is the proof that generic anisotropic structures of different order are contained in the Einstein theory. We reformulated the tensor and linear connection formalism for (pseudo) Riemannian spaces enables with N–connections and computed the horizonal–vertical splitting, with respect to anholonomic frames with associated N–connections, of the Einstein equations. The (pseudo) Riemannian spaces enabled with compatible anholonomic frame and associated N–connection structures and the metric being a solution of the Einstein equations were called as locally anisotropic spacetimes (la–spacetimes).

The next step was the definition of gauge field interactions on la–spacetimes. We have applied the bundle formalism and extended it to the case of bases being la–spacetimes and considered a pure geometric method of deriving the Yang–Mills equations for generic locally anisotropic gauge interactions, by genalizing the absolut differential calculus and dual forms symmetries for la–spacetimes.

The second key result was the proof by geometric methods that the Yang–Mills equations for a correspondingly defined Cartan connection in the bundle of affine frames on la–spacetimes are equivalent to the Einstein equations with anholonomic (N–connection) structures (the original Popov–Dikhin papers [23, 24] were for the locally isotropic spaces). The result was obtained by applying an auxiliary bilinear form on the tipical fiber because of degeneration of the Killing form for the affine groups. After projection on base spacetimes the dependence on auxiliar values is elliminated.

We analyzed also a variant of variational gauge locally anisotropic gauge theory by considering a minimal extension of the affine structural group to the de Sitter one, with a nonlinear realization for the gauge group as one was performed in a locally isotropic version in Tseytlin's paper [34]. If some former our works [33, 30] where devoted to extensions of some models of gauge gravity to generalized Lagrange and Finsler spaces, in this paper we demonstrated which manner

we could manage with anisotropies arrising in locally isotropic, but with anholonomic structures, variants of gauge gravity. Here it should be emphasized that anisotropies of different type (Finsler like, or more general ones) could be induced in all variants of gravity theories dealing with frame (tetrad, vierbiend, in four dimensions) fields and decompisitions of geometrical and physical objects in comonents with respect to such frames and associated N–connections. In a similar fashion anisotropies could arise under nontrivial reductions from higher to lower dimensions in Kaluza–Klein theories; in this case the N–connection should be treated as a splitting field modelling the anholonomic (anisotropic) character of some degrees of freedom.

The third basic result is the construction of a new class of solutions, with generic local anisotropy, of the Einstein equations. For simplicity, we defined these solutions in the framework of general relativity, but they can be removed to various variants of gauge and spinor gravity by using corresponding decompositions of the metric into the frame fields. We note that the obtained class of solutions also holds true for the gauge models of gravity which, in this paper, were constructed to be equivalent to the Einstein theory.

In explicit form we considered the metric ansatz

$$ds^2 = g_{\alpha\beta}\ du^\alpha du^\beta$$

when $g_{\alpha\beta}$ are parametrized by matrices of type

$$\left[\begin{array}{cccc} g_1 + {q_1}^2 h_3 + {n_1}^2 h_4 & 0 & q_1 h_3 & n_1 h_4 \\ 0 & g_2 + {q_2}^2 h_3 + {n_2}^2 h_4 & q_2 h_3 & n_2 h_4 \\ q_1 h_3 & q_2 h_3 & h_3 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 \end{array}\right] \tag{101}$$

with coefficients being some functions of necessary smooth class $g_i = g_i(x^j), q_i = q_i(x^j, t), n_i = n_i(x^j, t),\ h_a = h_a(x^j, t)$. Latin indices run respectively $i, j, k, ... = 1, 2$ and $a, b, c, ... = 3, 4$ and the local coordinates are denoted $u^\alpha = (x^i, y^3 = t, y^4)$, where t is treated as a timelike coordinate. A metric (101) can be diagonalized,

$$\delta s^2 = g_i(x^j) \left(dx^i\right)^2 + h_a(x^j, t)\left(\delta y^a\right)^2, \tag{102}$$

with respect to anholonomic frames (3) and (4), here we write down only the 'elongated' differentials

$$\delta t = dz + q_i(x^j, t) dx^i,\ \ \delta y^4 = dy^4 + n_i(x^j, t) dx^i.$$

The ansatz (101) was formally introduced in [32] in order to construct locally anisotropic black hole solutions; in this paper we applied it to cosmological la–spacetimes. In result, we get new metrics which describe locally anisotropic Friedman–Robertson–Walker like universes with the spherical symmetry deformed to that of rotation (ellongated and/or flattened) ellipsoid and torus. Such solutions are contained in general relativity: in the symplest diagonal form they are parametrized by distinguished metrics of type (102), given with respect to anholonomic bases, but could be also described equivalently with respect to a coordinate base by matrices of type (101). The topic of construction of cosmological models with generic spacetime and matter field distribution and fluctuation anisotropies is under consideration.

Now, we point the item of definition of reference frames in gravity theories: The form of basic field equations and fundamental laws in general relativity do not depend on choosing of coordinate systems and frame bases. Nevetheless, the problem of fixing of an adequate system of reference is also a very important physical task which is not solved by any dynamical equations but following some arguments on measuring of phyical observables, imposed symetry of interactions, types of horizons and singularities, and by taken into consideration the posed Cauchy problem. Having fixed a class of frame variables, the frame coefficients being presented in the Einstein equations, the type of constructed solution depends on the chosen holonomic or anholonomic frame structure. As a result one could model various forms of anisotropies in the framework of the Einsten theory (roughly, on (pseudo) Riemannian spacetimes with corresponding anholonomic frame structures it is possible to model Finsler like metrics, or more general ones with anisotropies). Finally, it should be noted that such questions on stability of obtained solutions, analysis of energy–momentum conditions should be performed in the simplest form with respect to the chosen class of anholonomic frames.

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